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THE VIRGINIA JOURNAL OF SCIENCE

A JOURNAL ISSUED QUARTERLY BY THE
VIRGINIA ACADEMY OF SCIENCE

Vol. 3, New Series JANUARY, 1952 No. 1

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VOL. 3, NEW SERIES JANUARY, 1952 NO. 1

THE VIRGINIA JOURNAL OF SCIENCE

PUBLISHED FOUR TIMES A YEAR: IN JANUARY, APRIL, JULY AND SEPTEMBER, BY THE VIRGINIA ACADEMY OF SCIENCE

Printed by the Commonwealth Press, Inc., Radford, Va.

CONTENTS

	Pages
The Distribution of the t and F Statistics for a Class of Non-Normal Populations—R. A. BRADLEY	1-32
A Comparison of the Gross Effects of Insulin Injection in Normal Mammals, Birds and Poikilotherms —D. R. H. GOURLEY	33-38
The Exoskeleton and Musculature of the Appendages of the First Three Abdominal Segments of *Cambarus longulus longulus* Girard (Decapoda Astacidae)—C. W. HART, JR.	39-48
On the Properties of the Multiple Comparisons Test—D. B. DUNCAN	49-67
Observations on the Egg Masses of Spotted Salamanders, *Ambystoma maculatum* (Shaw), the Williamsburg Area —JOHN THORNTON WOOD AND ROBERT HOLDEN WILKINSON	68-70
The Production of Hyperkeratosis by the Administration of a Lubricant—WILSON B. BELL	71-78
News and Notes	79-94

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Entered as second-class matter January 15, 1950, at the post office at Blacksburg, Virginia, under the Act of March 3, 1879. Subscription—$3.00 per volume. Published at Blacksburg, Va.

Mailed February 15, 1952

The Distribution of the t and F Statistics for a Class of Non-Normal Populations[1]

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INTRODUCTION

The study of errors of experiments involving small samples began with the fundamental work of "Student" (1908). A test of the hypothesis that the mean of a group of independent observations has a prespecified value was derived by considering the moments of the sample mean and of the sample standard error on the assumption that the observations came from a parent normal population. R. A. Fisher (1925, 1928) extended "Student's" work to obtain a test of the difference between the means of two groups of observations, and rigorously proved the results of "Student", at the same time introducing the use of the t statistic in this test. Fisher also obtained a test of the hypothesis that k groups of observations, assumed to come from normal populations with equal variances, have equal means. The z statistic used in that test is a simple monotone function of the more common statistic F used in the analysis of variance. This work is summarized by Fisher (1935) in an expository paper.

Uncertainties still exist regarding the practical validity of these tests when the assumptions implicit in their derivations are not satisfied. This matter is the subject of a sequence of papers by W. G. Cochran (1947), Churchill Eisenhart (1947), and M. S. Bartlett (1947). The assumptions in the analysis of variance are that

(i) The experimental errors are normally distributed;

(ii) The experimental errors have a common variance;

(iii) The experimental errors are independent; and

(iv) The group characteristics tested and the environmental effects are additive. It is with violations of assumption (i) that we shall deal in this paper.

There are many published papers on the subject of the non-normality of experimental errors, but the available information is unsatisfactory. The errors introduced in the significance levels of the t and F tests by

[1] Research supported in part by scholarships awarded by the National Research Council of Canada and by the Ontario Research Foundation, and presented to the Institute of Mathematical Statistics, Chicago, December 27, 1950.

the non-normality of parent universes require more detailed and comprehensive study. Such a study is necessary not so much because these tests are the appropriate ones for non-normal populations (and they are not usually best in these cases), but because it is desirable to use a common analysis which will give good results for a wide class of populations. Assurance that the results obtained are reliable is required.

Approximate methods of checking the validity of t and F tests are diverse. Transformations of the observed variates are discussed by Bartlett (1947). E. J. G. Pitman (1939) has given a general procedure for obtaining tests adapted to specified non-normal populations. Combinatorial and rank order tests are considered by Pitman (1937), Fisher (1947), Harold Hotelling and Margaret Pabst (1936), and by E. S. Pearson (1931). The principal discussion of limiting distributions of the t statistic is presented by Kai Lai Chung (1946) and his results are valid for large sample sizes. Verifications of the efficacy of the t test are obtained by experimental sampling by Pearson (1937), W. A. Shewhart and F. W. Winters (1928), and T. Eden and F. Yates (1933). P. R. Rider (1929), Victor Perlo (1933), and J. Laderman (1939) have discussed the exact distributions of t for rather trivial special cases and very small sample sizes. Bartlett (1935) and R. C. Geary (1936) have investigated errors in the significance levels of t by considering the first few terms of a Gram-Charlier series. Their results are approximate, depending on a knowledge of measures of skewness, and on neglecting powers of the sample size of lower order than its inverse and powers of the skewness of higher order than the first.

While the theoretical results so far obtained have not been particularly useful, still mathematical analysis appears to be the method by which more definite confirmation of the applicability of analysis of variance to non-normal populations may be derived. This paper was suggested by an asymptotic evaluation of corrections for non-normality at high significance levels by Hotelling (1947) and by an extension of his work by the present author (1950).

It is generally known that the cumulative distribution functions of t and of F may be written in the form of multiple integrals which depend on as many variables of integration as the number of observations in the sample and whose domains of integration depend on t and on F. We show that, for a large class of non-normal populations, the density functions of t and F may be written in terms of integrals whose domains of integration are independent of t and of F. These density functions in

turn are used to expand the cumulative distribution functions in powers of t^{-1} and of F^{-1}. The expansions are particularly useful at high significance levels and it is believed that these results represent a considerable improvement over previous investigations of the effects of non-normality.

In the following sections, derivations for the one-sample t test are given in detail. Only final results (derived by analogous methods) are shown for the two-sample t test and the F test.[2] Examples are developed for those cases in which the observations are drawn from the Cauchy and the "squared hyperbolic secant" distributions.

THE PROBABILITY DENSITY FUNCTION

Let $x_1, \cdots, x_N$ be a sample of N independent observations from a population with density function, $f(u)$. Without loss of generality, we use the t statistic to test the hypothesis that the mean of u is zero. In order to obtain a suitable expression for the cumulative distribution function of t, $G_N(t)$, it is required that $f(u)$ have the following properties:

(a) $f(u) > 0$ for all $-\infty < u < \infty$;

(b) $f(u)$ is a continuous function of u for $-\infty < u < \infty$. Now, using the familiar formula for t, we write

$$G_N(t) = \int \cdots \int_{\left(\sum_{i=1}^{N} x_i\right)\left\{N\sum_{i=1}^{N}(x_i - \bar{x})^2/(N-1)\right\}^{-1/2} < t} \prod_{i=1}^{N} f(x_i)\, dx_i. \tag{2.1}$$

If we use a prime to denote a column vector, we transform to new variables of integration by setting

$$y' = \Gamma x', \tag{2.2}$$

wherein we require that

$$\Gamma = \begin{bmatrix} 2^{-1/2} & -2^{-1/2} & 0 & \cdots \\ 6^{-1/2} & 6^{-1/2} & -2(6)^{-1/2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \{N(N-1)\}^{-1/2} & \{N(N-1)\}^{-1/2} & \{N(N-1)\}^{-1/2} & \{N(N-1)\}^{-1/2} \\ & \cdots & \{N(N-1)\}^{-1/2} & -\{(N-1)/N\}^{1/2} \\ N^{-1/2} & N^{-1/2} & N^{-1/2} & \\ & \cdots & N^{-1/2} & N^{-1/2} \end{bmatrix} \tag{2.3}$$

[2] A complete discussion is available (Bradley, 1949) in a dissertation in the library of the University of North Carolina at Chapel Hill.

Upon application of the transformation inverse to (2.2), we have

$$x_i = N^{-1/2}y_N - \{(i-1)/i\}^{1/2}y_{i-1} + \{i(i+1)\}^{-1/2}y_i + \cdots + \{(N-1)N\}^{-1/2}y_{N-1} \tag{2.4}$$

for all $i = 1, \cdots, N$ if it is understood that y_0 is taken to be zero. Since Γ is orthogonal, the domain of integration in (2.1) is simplified and it follows that

$$G_N(t) = \int \cdots \int_{y_N \left\{\sum_{i=1}^{N-1} y_i^2/(N-1)\right\}^{-1/2} < t} \prod_{i=1}^{N} f_i \prod_{i=1}^{N} dy_i . \tag{2.5}$$

We use f_i to simplify the writing of $f(x_i)$ when x_i is taken to be a function of the variables of integration.

By transforming to generalized spherical coordinates[3], we may reduce the domain of integration to one involving a single variable of integration. This procedure involves setting

$$\begin{aligned} y_N &= \rho \cos \Phi_1 \\ y_1 &= \rho \sin \Phi_1 \cos \Phi_2 \\ y_2 &= \rho \sin \Phi_1 \sin \Phi_2 \cos \Phi_3 \\ &\cdots \quad \cdots \quad \cdots \\ y_{N-2} &= \rho \sin \Phi_1 \sin \Phi_2 \sin \Phi_3 \cdots \sin \Phi_{N-2} \cos \Phi_{N-1} \\ y_{N-1} &= \rho \sin \Phi_1 \sin \Phi_2 \sin \Phi_3 \cdots \sin \Phi_{N-2} \sin \Phi_{N-1} . \end{aligned} \tag{2.6}$$

It is easy to verify that the transformation (2.6) will be one-to-one when we take the ranges of integration, 0 to π for $\Phi_1, \cdots, \Phi_{N-2}$, 0 to 2π for Φ_{N-1}, and 0 to ∞ for ρ. The jacobian is $\rho^{N-1} \sin^{N-2}\Phi_1 \sin^{N-3}\Phi_2 \cdots \sin \Phi_{N-2}$. In addition, if we set

$$t = (N-1)^{1/2} \cot \Theta, \tag{2.7}$$

the domain of integration in (2.5) is given by $\Theta < \Phi_1 \leqq \pi$. With the understanding that Θ is a function of t, we may now write

$$G_N(t) = \int_\Theta^\pi \sin^{N-2}\Phi_1 \left[\int_0^\infty \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \left\{ \prod_1^N f_i \right\} \cdot \rho^{N-1} \sin^{N-3}\Phi_2 \cdots \sin \Phi_{N-2}\, d\rho \prod_2^{N-1} d\Phi_i \right] d\Phi_1 . \tag{2.8}$$

[3] See, for example, Hotelling (1931).

It follows at once that

$$\frac{dG_N(t)}{d\Theta} = -\sin^{N-2}\Theta \cdot \left[\int_0^\infty \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \left\{\prod_1^N f_i\right\} \rho^{N-1} \sin^{N-3}\Phi_2 \cdots \sin\Phi_{N-2}\, d\rho \prod_2^{N-1} d\Phi_i\right]_{\Phi_1=\Theta} \tag{2.9}$$

The argument of f_i in (2.9) may be simplified by an inverse sequence of transformations. We begin by setting

$$u_i = (y_i/\sin\Phi_1) \mid_{\Phi_1=\Theta}, \qquad i = 1, \cdots, (N-1). \tag{2.10}$$

The jacobian is $(\rho^{N-2}\sin^{N-3}\Phi_2 \cdots \sin\Phi_{N-2})^{-1}$ and $\rho^2 = \sum_1^{N-1} u_i^2$. The final transformation is obtained by defining new variables,

$$\begin{aligned}
v_1 &= 2^{-1/2}u_1 + 6^{-1/2}u_2 + \cdots + \{(N-1)N\}^{-1/2}u_{N-1} \\
v_2 &= -2^{-1/2}u_1 + 6^{-1/2}u_2 + \cdots + \{(N-1)N\}^{-1/2}u_{N-1} \\
v_3 &= -2(6)^{-1/2}u_2 + \cdots + \{(N-1)N\}^{-1/2}u_{N-1} \\
&\cdots \quad \cdots \quad \cdots \\
v_{N-1} &= -(N-2)^{1/2}(N-1)^{-1/2}u_{N-2} + \{(N-1)N\}^{-1/2}u_{N-1}.
\end{aligned} \tag{2.11}$$

This transformation, although not orthogonal, has a simple jacobian. Let the matrix of the transformation be written as λ and then

$$\lambda\lambda' = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1/N \end{bmatrix}$$

so that $|\lambda| = N^{-1/2}$. The jacobian of the transformation from the u-coordinates to the v-coordinates is simply $N^{1/2}$. Further $v_1 + \cdots + v_{N-1} = (N-1)^{1/2}N^{-1/2}u_{N-1}$, and if (2.11) be written $v' = \lambda u'$, $vv' = u\lambda'\lambda u'$ and

$$\sum_1^{N-1} v_i^2 = \sum_1^{N-2} u_i^2 + N^{-1}u_{N-1}^2 = \sum_1^{N-1} u_i^2 - (N-1)N^{-1}u_{N-1}^2.$$

Then,

$$\sum_1^{N-1} u_i^2 = \sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2. \tag{2.12}$$

With the help of these transformations, and upon noting that the new variables are unrestricted on the real axis, we have

$$
\begin{aligned}
\frac{dG_N(t)}{d\Theta} &= -N^{1/2} \sin^{N-2}\Theta \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \\
&\quad f\left[\sin\Theta\left\{N^{-1/2}\sqrt{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2}\cot\Theta - \sum_1^{N-1} v_i\right\}\right] \\
&\quad \cdot \prod_{i=1}^{N-1} f\left[\sin\Theta\left\{N^{-1/2}\sqrt{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2}\cot\Theta + v_i\right\}\right] \\
&\quad \cdot \sqrt{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2} \prod_1^{N-1} dv_i .
\end{aligned}
\tag{2.13}
$$

$G'_N(t) = dG_N(t)/dt$ may be derived from (2.13) when we recall that $t = (N-1)^{1/2}\cot\Theta$ from whence $\sin\Theta = \{1 + t^2/(N-1)\}^{-1/2}$ and $G'_N(t) = \dfrac{d}{d\Theta}G_N(t)\cdot\dfrac{d\Theta}{dt}$ with $d\Theta/dt = -(N-1)^{-1/2}\sin^2\Theta$. Hence,

$$
\begin{aligned}
G'_N(t) &= N^{1/2}(N-1)^{-1/2}\{1 + t^2/(N-1)\}^{-N/2} \\
&\quad \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left[\{1 + t^2/(N-1)\}^{-1/2}\right. \\
&\quad \left.\cdot\left\{N^{-1/2}(N-1)^{-1/2}\sqrt{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2}\, t - \sum_1^{N-1} v_i\right\}\right] \\
&\quad \cdot \prod_{i=1}^{N-1} f\left[\{1 + t^2/(N-1)\}^{-1/2}\right. \\
&\quad \left.\cdot\left\{N^{-1/2}(N-1)^{-1/2}\sqrt{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2}\, t + v_i\right\}\right] \\
&\quad \cdot \sqrt{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2} \prod_1^{N-1} dv_i .
\end{aligned}
\tag{2.14}
$$

We have now obtained an expression for the probability density function of t for the large class of populations satisfying conditions (a) and (b). When $f(u)$ is the normal density, it is very easy to show that (2.14) gives the familiar "Student" distribution. When $f(u)$ is not the normal density, it may be possible to evaluate $G'_N(t)$ directly in rare cases and for small sample sizes. More difficult cases may be treated by the method of the following section.

A SERIES REPRESENTATION OF THE CUMULATIVE DISTRIBUTION FUNCTION

$G_N(t)$ will be expanded in a Taylor Series in powers of $1/t$. To do this, it is necessary that $f(u)$ have as high order continuous derivatives as terms used in the expansion. We then may require the condition,

(c) $d^k f(u)/du^k$ is a continuous function of u for $-\infty < u < \infty$ and $k = 1, 2, 3, \cdots$.

It will be convenient to write

$$P(v) = N^{-1/2}\left\{\sum_1^{N-1} v_i^2 + \left(\sum_1^{N-1} v_i\right)^2\right\}^{1/2} \quad \text{and} \quad n = N - 1 \tag{3.1}$$

in the expression for $dG_N(t)/d\Theta$ in (2.13). In that formula, we set

$$z_i = v_i \mid \cos\Theta \mid, \qquad i = 1, \cdots, n, \tag{3.2}$$

and this is a permissible transformation so long as $\Theta \neq \pi/2$. The jacobian is simply $\mid \cos\Theta \mid^{-n}$. We also define

$$\Phi = \mid \tan\Theta \mid = n^{1/2} \mid t \mid^{-1}. \tag{3.3}$$

When Θ is between 0 and $\pi/2$, $d\Theta/d\Phi = \cos^2\Theta$ and t is positive; when Θ is between $\pi/2$ and π, $d\Theta/d\Phi = -\cos^2\Theta$ and t is negative. Upon multiplying (2.13) by $d\Theta/d\Phi$ and applying (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned} dG_N(t)/d\Phi = -N\Phi^{N-2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_1^n f\{P(z) + \Phi z_i\}\right] \\ \cdot f\left\{P(z) - \Phi \sum_1^n z_i\right\} P(z) \prod_1^n dz_i, \end{aligned} \tag{3.4}$$

for positive values of t, and

$$\begin{aligned} dG_N(t)/d\Phi = N\Phi^{N-2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_1^n f\{-P(z) + \Phi z_i\}\right] \\ \cdot f\left\{-P(z) - \Phi \sum_1^n z_i\right\} P(z) \prod_1^n dz_i, \end{aligned} \tag{3.5}$$

for t negative.

In the following discussion we shall use the Taylor Expansion of $G_N(t)$ in powers of Φ about the point $\Phi = 0$. The form (3.4) will be considered in detail and the result for (3.5) may be written down in a similar way.

$dG_N(t)/d\Phi$ may be taken to be the product of two functions of Φ and

successive derivatives may be obtained by the application of Leibnitz' rule. We are concerned with the derivatives of

$$A(\Phi) = \Phi^{N-2}, \tag{3.6}$$

and

$$B(\Phi) = \left[\prod_{i=1}^{n} f\{P(z) + \Phi z_i\}\right] f\left\{P(z) - \Phi \sum_{1}^{n} z_i\right\} \tag{3.7}$$

evaluated at $\Phi = 0$. The required derivatives of $A(\Phi)$ vanish except for

$$d^{N-2}A(\Phi)/d\Phi^{N-2} = \Gamma(n). \tag{3.8}$$

A generalization of Leibnitz' rule permits us to write

$$d^k B(\Phi)/d\Phi^k \,|_{\Phi=0} = \sum_{\alpha_0+\cdots+\alpha_n=k} \frac{k!}{\alpha_0! \cdots \alpha_n!} (-\Sigma z_i)^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n} f^{(\alpha_0)}\{P(z)\} \cdots f^{(\alpha_n)}\{P(z)\}. \tag{3.9}$$

By means of (3.5), (3.8) and (3.9) and, again, by the application of Leibnitz' rule, we may state the result that

$$\begin{aligned} G_N(t) = {} & G_N(t)\,|_{\Phi=0} - \frac{N\Phi^n}{n!}\binom{n-1}{0}\Gamma(n)\int_{-\infty}^{\infty}\cdots \\ & \int_{-\infty}^{\infty} B(\Phi)\,|_{\Phi=0} P(z)\prod_{1}^{n} dz_i - \cdots - \frac{N\Phi^{n+k}}{(n+k)!}\binom{n+k-1}{k}\Gamma(n)\int_{-\infty}^{\infty} \\ & \cdots \int_{-\infty}^{\infty} \frac{d^k B(\Phi)}{d\Phi^k}\,|_{\Phi=0} P(z)\prod_{1}^{n} dz_i - \cdots . \end{aligned} \tag{3.10}$$

If we recall that $\Phi = n^{1/2} \mid t \mid^{-1}$, when $t > 0$, $G_N(t)\,|_{\Phi=0} = 1$, and

$$\begin{aligned} G_N(t) = {} & 1 - \frac{Nn^{n/2}}{n!}\Gamma(n)\binom{n-1}{0}t^{-n}\int_{-\infty}^{\infty} \\ & \cdots \int_{-\infty}^{\infty} B(\Phi)|_{\Phi=0} P(z)\prod_{1}^{n} dz_i \\ & - \cdots - \frac{Nn^{(n+k)/2}}{(n+k)!}\Gamma(n)\binom{n+k-1}{k}t^{-(n+k)}\int_{-\infty}^{\infty} \\ & \cdots \int_{-\infty}^{\infty} \frac{d^k B(\Phi)}{d\Phi^k}\,|_{\Phi=0} P(z)\prod_{1}^{n} dz_i - \cdots , \end{aligned} \tag{3.11}$$

for positive values of t.

When t is negative, we need only define.

$$B^*(\Phi) = f\left\{-P(z) - \Phi \sum_1^n z_i\right\} \prod_1^n f\{-P(z) + \Phi z_i\}, \tag{3.12}$$

and use (3.5) in place of (3.4). Then, when $t < 0$, $G_N(t) \mid_{\Phi=0} = 0$, and the expression equivalent to (3.11) for this case is

$$\begin{aligned} G_N(t) = {} & \frac{Nn^{n/2}}{n!}\Gamma(n)\binom{n-1}{0} \mid t \mid^{-n} \int_{-\infty}^{\infty} \\ & \cdots \int_{-\infty}^{\infty} B^*(\Phi) \mid_{\Phi=0} P(z) \prod_1^n dz_i \\ & + \cdots + \frac{Nn^{(n+k)/2}}{(n+k)!}\Gamma(n)\binom{n+k-1}{k} \mid t \mid^{-(n+k)} \int_{-\infty}^{\infty} \\ & \cdots \int_{-\infty}^{\infty} \frac{d^k B^*(\Phi)}{d\Phi^k} \mid_{\Phi=0} P(z) \prod_1^n dz_i + \cdots . \end{aligned} \tag{3.13}$$

These expansions of $G_N(t)$ hold promise of rapid convergence for large absolute values of t, and afford a method of obtaining at least a few terms of the distribution function of t when the population density function, $f(u)$, satisfies conditions (a), (b), and (c). We now develop a systematic method of evaluating the terms of these series.

THE REDUCTION OF THE COEFFICIENTS

The expansion of $G_N(t)$ depends on the integral of $d^kB(\Phi)/d\Phi^k \mid_{\Phi=0} P(z)$ over unrestricted values of the real variables of integration. The k^{th} derivative of $B(\Phi)$ may be expressed in two parts, one of which is independent of the density, $f(u)$, and hence common to all functions $G_N(t)$, and another much simpler expression depending on the particular population under consideration. The first step in such a procedure is to note that $P(z)$ and $\sum_1^n z_i$ are symmetric functions of $z_1, \cdots, z_n$. Hence, from (3.9),

$$\frac{d^kB(\Phi)}{d\Phi^k} \mid_{\Phi=0} \doteq k!\, n! \sum_{\substack{\alpha_0+\cdots+\alpha_n=k \\ \alpha_1 \leqq \cdots \leqq \alpha_n}} \frac{(-\Sigma z_i)^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n}}{n_1! \cdots n_p!} \prod_{j=0}^n \frac{f^{(\alpha_j)}\{P(z)\}}{\alpha_j!} \tag{4.1}$$

where $\alpha_1, \cdots, \alpha_n$ are assumed to take on p distinct values, the first of which occurs n_1 times, the second n_2 times and so on.[4]

[4] We use the symbol $\doteq$ to read "has the same integral as" with respect to the variables present. Thus, if $u = \lambda v$, $\int_{-\infty}^{\infty} g(u)\, du = \int_{-\infty}^{\infty} g(\lambda v)\lambda\, dv$, and $g(u) \doteq \lambda g(\lambda v)$.

Many linear transformations may be used to transform $P(z)$ to a function of a sum of squares, but, in order to take advantage of the ordering of the exponents of $z_1, \cdots, z_n$, we must choose a transformation such that the complexity of the expressions for the z's in terms of the new variables decreases from z_1 to z_n. The transformation best suited to this plan is one we have already used (2.2, 2.3). It is only necessary to replace N by n in the matrix, Γ. If the new variables are taken as w's, we may write

$$z_i = n^{-1/2}w_n - i^{-1/2}(i-1)^{1/2}w_{i-1} + \{i(i+1)\}^{-1/2}w_i + \cdots + \{(n-1)n\}^{-1/2}w_{n-1}, \qquad i = 1, \cdots, n. \tag{4.2}$$

Now,

$$P(z) = \left\{\sum_1^n z_i^2 + \left(\sum_1^n z_i\right)^2\right\}^{1/2} N^{-1/2} = \left\{\sum_1^n w_i^2 + nw_n^2\right\}^{1/2} N^{-1/2}.$$

If we set $w_n' = N^{1/2}w_n$, we have jacobian $N^{-1/2}$, and upon replacing w_n by $w_n'/N^{1/2}$ and dropping the prime, the expressions required are

$$z_i = (nN)^{-1/2}w_n - i^{-1/2}(i-1)^{1/2}w_{i-1} + \{i(i+1)\}^{-1/2}w_i + \cdots + \{(n-1)n\}^{-1/2}w_{n-1}, \qquad i = 1, \cdots, n,$$

$$\sum_1^n z_i = w_n(n/N)^{1/2}, \tag{4.3}$$

$$P(z) = \left(\sum_1^n w_i^2/N\right)^{1/2}.$$

For brevity, let us write z_i in terms of $w_1, \cdots, w_n$ as $z_i(w)$.

A transformation to generalized spherical coordinates is again specified. A first step sets

$$w_i = \rho\epsilon_i, \qquad i = 1, \cdots, n, \quad \text{with} \quad \sum_1^n \epsilon_i^2 = 1. \tag{4.4}$$

The second step requires that we take

$$\begin{aligned}
\epsilon_1 &= \sin\Theta_n \sin\Theta_{n-1} \cdots \sin\Theta_2 \\
&\cdots \quad \cdots \quad \cdots \\
\epsilon_i &= \sin\Theta_n \sin\Theta_{n-1} \cdots \sin\Theta_{i+1}\cos\Theta_i, \qquad i = 2, \cdots, (n-1) \\
&\cdots \quad \cdots \quad \cdots \\
\epsilon_n &= \cos\Theta_n.
\end{aligned} \tag{4.5}$$

The jacobian of the transformation is

$$\rho^{n-1}J(\Theta_2, \cdots, \Theta_n) \tag{4.6}$$

with $J(\Theta_2, \cdots, \Theta_n) = \sin^{n-2}\Theta_n \sin^{n-3}\Theta_{n-1} \cdots \sin\Theta_3$, and the ranges of integration are 0 to π for $\Theta_3, \cdots, \Theta_n$, and 0 to 2π for Θ_2.

Since the transformation (4.3) is linear, $z_i^{\alpha_i}$ is a homogeneous function of $w_1, \cdots, w_n$ of degree α_i, and we may write $z_i^{\alpha_i} = \rho^{\alpha_i}\{z_i(\epsilon)\}^{\alpha_i}$. In terms of the new variables (4.1) may be written

$$\left.\frac{d^k B(\Phi)}{d\Phi^k}\right|_{\Phi=0}$$

$$\doteq \frac{k!\,n!}{\sqrt{N}} \sum_{\substack{\alpha_0+\cdots+\alpha_n=k \\ \alpha_1\leq\cdots\leq\alpha_n}} \frac{z_1^{\alpha_1}(\epsilon)\cdots z_n^{\alpha_n}(\epsilon)\left\{-\epsilon_n\sqrt{\frac{n}{N}}\right\}^{\alpha_0}}{n_1!\cdots n_p!} J(\Theta_2, \cdots, \Theta_n) \cdot \rho^{n+k-1}\prod_{j=0}^{n}\frac{f^{(\alpha_j)}(\rho/\sqrt{N})}{\alpha_j!} \tag{4.7}$$

where the accumulated jacobians are included.

It is well to note at this point that we have divided each term in the summation into two parts, one of which is a function of ρ alone, and therefore an even function of $w_1, \cdots, w_n$, whereas the other is independent of ρ and is, excluding $J(\Theta_2, \cdots, \Theta_n)$, a polynomial of degree k in terms of $\epsilon_1, \cdots, \epsilon_n$. The previous analogous part was then a polynomial of degree k in $w_1, \cdots, w_n$, and when k is odd, at least one of $w_1, \cdots, w_n$ appeared as an odd function in every term of

$$z_1^{\alpha_1}(w)\cdots z_n^{\alpha_n}(w)\left\{-w_n\sqrt{\frac{n}{N}}\right\}^{\alpha_0}.$$

Since $P(z)$ is an even function of the w's, the integrals of $d^kB(\Phi)/d\Phi^k|_{\Phi=0}$ $P(z)$ are integrals of odd functions whenever k is odd and then vanish independent of the form of $f(u)$. As a further observation, when k is even, certain terms of

$$z_1^{\alpha_1}(w)\cdots z_n^{\alpha_n}(w)\left\{-w_n\sqrt{\frac{n}{N}}\right\}^{\alpha_0}$$

will contain one or more of the variates $w_1, \cdots, w_n$ with an odd exponent and such terms may again be considered to be odd and may be omitted. This property carries over to $\epsilon_1, \cdots, \epsilon_n$ and we need only consider terms which consist of products of $\epsilon_1, \cdots, \epsilon_n$ all with even exponents. For example, when $k = 4$, we consider only terms containing ϵ_i^4 and $\epsilon_i^2\epsilon_j^2$. Elaboration of these simplifying properties will later be in

order but the discussion is included here since they are more easily detected in terms of $w_1, \cdots, w_n$ and carry over to $\epsilon_1, \cdots, \epsilon_n$.

Let us define

$$C_N(\epsilon, k; a_0, \cdots, a_n) = \sum_A \frac{n!}{n_1! \cdots n_p!} \left\{ -\epsilon_n \sqrt{\frac{n}{N}} \right\}^{\alpha_0} \cdot J(\Theta_2, \cdots, \Theta_n) \prod_{j=1}^{n} z_j^{\alpha_j}(\epsilon) \tag{4.8}$$

where $a_0, \cdots, a_n$ is a combination of integers such that $a_0 + \cdots + a_n = k$, and the summation over the set A represents a summation over the fixed combination with α_0 taking each distinct value of the combination once and only once with $\alpha_1 \leqq \cdots \leqq \alpha_n$ taking the remaining values after α_0 has been determined. p refers to the number of distinct values in this remaining set with the first occurring n_1 times and so on. $C_N(\epsilon, k; a_0, \cdots, a_n)$ is then well defined. We shall further write

$$D_N(\rho, k; a_0, \cdots, a_n) = \rho^{n+k} \prod_{j=0}^{n} \{ f^{(a_j)}(\rho/\sqrt{N})/a_j! \} \tag{4.9}$$

and note that permutations of $a_0, \cdots, a_n$ leave $D_N(\rho, k; a_0, \cdots, a_n)$ invariant. Using (4.7), we have

$$\left. \frac{d^k B(\Phi)}{d\Phi^k} \right|_{\Phi=0} P(z) \doteq N^{-1} k! \sum_{\substack{\text{comb.} \\ a_0+\cdots+a_n=k}} C_N(\epsilon, k; a_0, \cdots, a_n) D_N(\rho, k; a_0, \cdots, a_n), \tag{4.10}$$

where the summation has now been reduced to one over the combinations of integers $a_0 + \cdots + a_n = k$ instead of one over the permutations of the same set of integers. The expression for $P(z)$, given in (4.3), has been included in the right hand side of (4.10).

The integral of $d^k B(\Phi)/d\Phi^k|_{\Phi=0} P(z)$ may be writen as the product of two integrals. It is only necessary to define

$$D_{IN}(k; a_0, \cdots, a_n) = \int_0^\infty D_N(\rho, k; a_0, \cdots, a_n)\, d\rho \tag{4.11}$$

and

$$C_{IN}(k; a_0, \cdots, a_n) = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} C_N(\epsilon, k; a_0, \cdots, a_n)\, d\Theta_n \cdots d\Theta_2. \tag{4.12}$$

Then,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d^k B(\Phi)}{d\Phi^k}\bigg|_{\Phi=0} P(z) \prod_1^n dz_i = \frac{k!}{N} \sum_{\substack{\text{comb.} \\ a_0+\cdots+a_n=k}} C_{IN}(k; a_0, \cdots, a_n) \cdot D_{IN}(k; a_0, \cdots, a_n). \tag{4.13}$$

In exactly the same way as that described above, it is possible to write

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d^k B^*(\Phi)}{d\Phi^k}\bigg|_{\Phi=0} P(z) \prod_1^n dz_i = \frac{k!}{N} \sum_{\substack{\text{comb.} \\ a_0+\cdots+a_n=k}} C_{IN}(k; a_0, \cdots, a_n) \cdot D^*_{IN}(k; a_0, \cdots, a_n) \tag{4.14}$$

where

$$D^*_N(\rho, k; a_0, \cdots, a_n) = \rho^{n+k} \prod_{j=0}^{n} \{f^{(\alpha_j)}(-\rho/\sqrt{N})/a_j!\} \tag{4.15}$$

and

$$D^*_{IN}(k; a_0, \cdots, a_n) = \int_0^\infty D^*_N(\rho, k; a_0, \cdots, a_n)\, d\rho. \tag{4.16}$$

THE EVALUATION OF $C_{IN}(k; a_0, \cdots, a_n)$

$C_{IN}(k; a_0, \cdots, a_n)$ need only be evaluated once for each set of values $a_0 + \cdots + a_n = k$ since it is independent of the particular density considered. As previously indicated, C_{IN} vanishes for all odd values of k and we then need examine only the even values. In order to obtain C_{IN} it is first necessary to write C_N as an explicit function of $\epsilon_1, \cdots, \epsilon_n$ and to proceed with the transformation (4.5). We omit all terms of C_N which contain an odd power of any ϵ_i, and, furthermore, from symmetry, $\epsilon_1^{2\beta_1} \cdots \epsilon_n^{2\beta_n}$ may be replaced by $\epsilon_1^{2\beta_1'} \cdots \epsilon_n^{2\beta_n'}$, where $\beta_1', \cdots, \beta_n'$ is a rearrangement of $\beta_1, \cdots, \beta_n$ with $\beta_1' \leqq \cdots \leqq \beta_n'$. Thus, for $k = 4$, say, C_N may be reduced to a linear function of ϵ_n^4 and $\epsilon_n^2\epsilon_{n-1}^2$ only.

When $k = 0$, $C_N(\epsilon, 0; 0, \cdots, 0) = J(\Theta_2, \cdots, \Theta_n)$ and $C_{IN}(0; 0, \cdots, 0) = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{n-2}\Theta_n \sin^{n-3}\Theta_{n-1} \cdots \sin\Theta_3\, d\Theta_n \cdots d\Theta_2$. If we recall that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\Phi \cos^{2n-1}\Phi\, d\Phi$, it is easy to show that

$$C_{IN}(0; 0, \cdots, 0) = 2\pi^{n/2}/\Gamma(n/2) = S_{n-1}. \tag{5.1}$$

This defines the constant, S_{n-1}.

The calculations of C_{IN} are tedious but straight-forward. We illustrate by evaluating $C_{IN}(4; 2, 1, 1, 0, \cdots, 0)$. $C_N(\epsilon, 4; 2, 1, 1, 0, \cdots, 0)$ is obtained by substituting for $z_j(\epsilon)$ using (4.3), which defines $z_j(w)$, in (4.8). Upon applying the rules of procedure given at the beginning of this section, we have

$$C_N(\epsilon, 4; 2, 1, 1, 0, \cdots, 0) \doteq J(\Theta_2, \cdots, \Theta_n)\left\{\epsilon_n^4\left(\frac{n^3 - n^2 - 4n + 4}{2nN}\right) - \epsilon_n^2\epsilon_{n-1}^2\left(\frac{n^4 - n^2 - 12n + 12}{2nN}\right)\right\}.$$

$C_{IN}(\epsilon, 4; 2, 1, 1, 0, \cdots, 0)$ will depend on two integrals for which the variables of integration are $\Theta_2, \cdots, \Theta_n$. It is necessary to use (4.5) to express ϵ_n and ϵ_{n-1} as functions of the required angles. Integration depends on the definition of the Beta integral and it can be shown that

$$\int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \epsilon_n^4(\Theta_2, \cdots, \Theta_n) J(\Theta_2, \cdots, \Theta_n)\, d\Theta_n \cdots d\Theta_2 = 3S_{n-1}/\{n(n+2)\}$$

and

$$\int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \epsilon_n^2(\Theta_2, \cdots, \Theta_n)\epsilon_{n-1}^2(\Theta_2, \cdots, \Theta_n) J(\Theta_2, \cdots, \Theta_n)\, d\Theta_n \cdots d\Theta_2 = S_{n-1}/\{n(n+2)\}.$$

Then,

$$(5.2) \qquad C_{IN}(4; 2, 1, 1, 0, \cdots, 0) = \frac{-S_{n-1}}{N(n+2)}\left(\frac{n^2 - 3n + 2}{2}\right).$$

Values of the C_{IN} coefficients are listed below for values of k up to six. They are sufficient for obtaining the first four terms of the expansions of $G_N(t)$, for populations whose densities satisfy the mild conditions imposed, when combined with the coefficients D_{IN} which alone depend on the population density.

We list[5] the coefficients:

$$(5.3) \qquad k = 0 \qquad \delta_0 = S_{n-1}$$

$$C_{IN}(0; 0, \cdots, 0) = \delta_0$$

$$k = 2 \qquad \delta_2 = S_{n-1}$$

$$C_{IN}(2; 2, 0, \cdots, 0) = \delta_2$$

[5] See footnote 2.

$$C_{IN}(2; 1, 1, 0, \cdots ,0) = -\tfrac{1}{2}\delta_2$$

$$k = 4 \qquad \delta_4 = S_{n-1}/\{N(n+2)\}$$

$$C_{IN}(4; 4, 0, \cdots ,0) = 3n\delta_4$$

$$C_{IN}(4; 3, 1, 0, \cdots ,0) = -3n\delta_4$$

$$C_{IN}(4; 2, 2, 0, \cdots ,0) = \tfrac{1}{2}(n^2+2)\delta_4$$

$$C_{IN}(4; 2, 1, 1, 0, \cdots ,0) = -\tfrac{1}{2}(n^2-3n+2)\delta_4$$

$$C_{IN}(4; 1, 1, 1, 1, 0, \cdots ,0) = \tfrac{1}{8}(n^2-3n+2)\delta_4$$

$$k = 6 \qquad \delta_6 = S_{n-1}/\{N^2(n+2)(n+4)\}$$

$$C_{IN}(6; 6, 0, \cdots ,0) = 15n^2\delta_6$$

$$C_{IN}(6; 5, 1, 0, \cdots ,0) = -15n^2\delta_6$$

$$C_{IN}(6; 4, 2, 0, \cdots ,0) = 3n(n^2+4)\delta_6$$

$$C_{IN}(6; 4, 1, 1, 0, \cdots ,0) = -\tfrac{3}{2}n(n^2-5n+4)\delta_6$$

$$C_{IN}(6; 3, 3, 0, \cdots ,0) = -\tfrac{3}{2}(3n^2+2)\delta_6$$

$$C_{IN}(6; 3, 2, 1, 0, \cdots ,0) = -3(n^3-3n^2+4n-2)\delta_6$$

$$C_{IN}(6; 3, 1, 1, 1, 0, \cdots ,0) = \tfrac{1}{2}(3n^3-11n^2+12n-4)\delta_6$$

$$C_{IN}(6; 2, 2, 2, 0, \cdots ,0) = \tfrac{1}{6}(n^4-n^3+6n^2-14n+8)\delta_6$$

$$C_{IN}(6; 2, 2, 1, 1, 0, \cdots ,0) = -\tfrac{1}{4}(n^4-7n^3+24n^2-38n+20)\delta_6$$

$$C_{IN}(6; 2, 1, 1, 1, 1, 0, \cdots ,0) = \tfrac{1}{8}(n^4-10n^3+35n^2-50n+24)\delta_6$$

$$C_{IN}(6, 1, 1, 1, 1, 1, 1, 0, \cdots ,0) = -\tfrac{1}{48}(n^4-10n^3+35n^2-50n+24)\delta_6$$

A partial check on these coefficients has been made. This was accomplished by taking $f(u)$ as the normal density function and computing the coefficients, D_{IN}. Values of D_{IN} and C_{IN} were substituted in (4.13) and the values of these integrals were in turn used on the expressions for $G_N(t)$ given in (3.11) and (3.13). The series were checked by expanding the "Student" density in powers of t^{-1} and integrating that power series term by term with respect to t over the range $-\infty$ to t_0, say. This should give us $G_N(t_0)$ and in fact the first four terms are identical with the results obtained by the general method of this paper when t is replaced by t_0.

ILLUSTRATIVE EXAMPLES

We now consider applications of the series expansion of the cumulative distribution function of t to two non-normal populations. They have the Cauchy and the "squared hyperbolic secant" densities for which $f(u)$ takes the forms

$$(6.1) \qquad (C)\ \pi^{-1}(1+u^2)^{-1} \quad \text{and} \quad (S)\ \frac{1}{4}\operatorname{sech}^2 u/2$$

respectively. Standard forms may be used since the distribution of t is independent of location and scale parameters.

We first consider the Cauchy density and require expressions for its derivatives. A result of J. Bertrand (1864) expresses the general derivative of $\tan^{-1}u$ in compact form and is useful here since $f(u) = \pi^{-1}\,\partial \tan^{-1}u/du$. Bertrand's result is in slight error but, with the proper correction,

$$(6.2\ C) \qquad f^{(\gamma)}(u) = \pi^{-1}\gamma!\,(-1)_R \sin^{\gamma+1} y \sin(\gamma+1)y,$$

where $\sin y = (1+u^2)^{-1/2}$ and $\cos y = u(1+u^2)^{-1/2}$. Upon substituting u for $\rho/\sqrt{N}$ in (4.11), with the help of (4.9), we have D_{IN} expressed in terms of the derivatives of $f(u)$. Then, we use the simple transformation, $u = \cot y$ and substitute the values, (6.2 C). It follows that

$$(6.3\ C) \qquad D_{IN}(k; a_o, \cdots, a_n) = \pi^{-N}(-1)^k N^{(N+k)/2} \int_0^{\pi/2} \cos^{n+k} y \operatorname{cosec} y \cdot \left\{\prod_{i=0}^{n} \sin(a_i+1)y\right\} dy.$$

These integrals are evaluated by expanding the trigonometric functions of the multiple angles in terms of $\sin y$ and $\cos y$. The values required are tabled below.

Derivatives of the squared hyperbolic secant density may be obtained simply by repeated differentiation. Even-order derivatives are polynomials in $\operatorname{sech}^2 u/2$; odd-order derivatives are products of such polynomials and $\tanh u/2$. Integrals, $D_{I\nu}$, vanish unless the sum of the orders of the derivatives is even. Thus, when u replaces $\rho/\sqrt{N}$ in these integrals, $\tanh u/2$ appears with an even exponent and such functions may be expressed in terms of $\operatorname{sech}^2 u/2$. It follows that all D_{IN} depend on integrals of the form,

$$(6.4\ S) \qquad I(p, q) = \int_0^\infty x^p \operatorname{sech}^{2q} x/2\, dx = 2^q \int_0^\infty x^p (1+\cosh x)^{-q}\, dx,$$

where p is a positive integer or zero and q is a positive integer.

$I(p, q)$ is not in general an elementary function; however the following recursion relations are useful:

$$I(p, q) = \frac{2(q-1)}{(2q-1)} I(p, q-1) - \frac{2p(p-1)}{(q-1)(2q-1)} I(p-2, q-1), \qquad p \geqq 2, \tag{6.5 S}$$

and

$$I(p, q) = \frac{2(q-1)}{(2q-1)} I(p, q-1) - \frac{2^q p}{(2q-1)} \int_0^\infty x^{p-1} (1 + \cosh x)^q \sinh x \, dx. \tag{6.6 S}$$

The second relation is used when $p = 1$ and its last term is then easily

TABLE I

VALUES OF $I(p, q)$

p^q	2	3	4	5	6	7
1	1.18173					
2		0.84963				
3		1.24309	0.41099			
4			0.86625	0.41994		
5		4.56605	1.54594	0.91750		
6			4.13962	2.23589	1.57450	
7		29.25208	6.80901	2.44526	1.82169	
8			17.08400	8.28641	2.98001	0.48996
9				11.91614	4.43072	2.57293

integrable. Repeated use of the reduction formulae will terminate with one or more of the integrals[6]:

$$\int_0^\infty x^{2a} (1 + \cosh x)^{-1} \, dx = 2(2^{2a-1} - 1)\pi^{2a} B_a,$$

$$\int_0^\infty x^{2a+1} (1 + \cosh x)^{-1} \, dx = 4\Gamma(2a+2) \left\{ \frac{2^{2a} - 1}{2^{2a+1} - 1} \right\} \sum_{j=0}^{\infty} (2j+1)^{-(2a+1)}, \qquad a \geqq 1, \tag{6.7 S}$$

[6] B_a is the a^{th} Bernoulli number, and, of the various definitions of these numbers, the one used here is that B_a is the coefficient of $x^{2a}/(2a)!$ in the expansion of $xe^x/(e^x - 1)$. $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$.

$$\int_0^\infty x(1 + \cosh x)^{-1}\,dx = 2\log_e 2, \qquad \text{and}$$

$$\int_0^\infty (1 + \cosh x)^{-b}\,dx = 2^{b-1}\beta(b, b).$$

The values of $I(p, q)$, which we shall require, are shown in Table I.

The coefficients, D_{IN}, are given for both the Cauchy and the squared hyperbolic secant densities in order in (6.9 CS). We abbreviate by setting

(6.8 CS) $$\beta\left(\frac{n+i}{2}, \frac{n+j}{2}\right) = b(i, j) \quad \text{and} \quad I(n+i, n+j) = d(i, j).$$

Then we list:

(6.9 CS) $\underline{k = 0}$ $\gamma_0 = N^{N/2}\pi^{-N}$, $\Delta_0 = N^{N/2}2^{-2N}$.

$D_{IN}(0; 0, \cdots ,0) = \gamma_0\, b(1, 1)/2; \quad \Delta_0\, d(0, 1).$

$\underline{k = 2}$ $\gamma_2 = N^{(N+2)/2}\pi^{-N}$, $\Delta_2 = N^{(N+2)/2}2^{-2N}/2!$

$D_{IN}(2; 2, 0, \cdots ,0) = \gamma_2\{\tfrac{3}{2}b(3, 1) - 2b(3, 3)\};$
$\Delta_2\{d(2, 1) - \tfrac{3}{2}d(2, 2)\}.$

$D_{IN}(2; 1, 1, 0, \cdots ,0) = 2\gamma_2\, b(5, 1); 2\Delta_2\{d(2, 1) - d(2, 2)\}.$

$\underline{k = 4}$ $\gamma_4 = N^{(N+4)/2}\pi^{-N}$, $\Delta_4 = N^{(N+4)/2}2^{-2N}/4!$

$D_{IN}(4; 4, 0, \cdots ,0) = \gamma_4\{\tfrac{5}{2}b(5, 1) - 10b(5, 3) + 8b(5, 5)\};$
$\Delta_4\{d(4, 1) - \tfrac{15}{2}d(4, 2) + \tfrac{15}{2}d(4, 3)\}.$

$D_{IN}(4; 3, 1, 0, \cdots ,0) = \gamma_4\{8b(9, 1) - 4b(7, 1)\}; \Delta_4\{4d(4, 1)$
$- 16d(4, 2) + 12d(4, 3)\}.$

$D_{IN}(4; 2, 2, 0, \cdots ,0) = \gamma_4\{\tfrac{9}{2}b(5, 1) - 12b(5, 3) + 8b(5, 5)\};$
$\Delta_4\{6d(4, 1) - 18d(4, 2) + \tfrac{27}{2}d(4, 3)\}.$

$D_{IN}(4; 2, 1, 1, 0, \cdots ,0) = \gamma_4\{6b(7, 1) - 8b(7, 3)\};$
$\Delta_4\{12d(4, 1) - 30d(4, 2) + 18d(4, 3)\}.$

$D_{IN}(4; 1, 1, 1, 1, 0, \cdots ,0) = 8\gamma_4 b(9, 1); \Delta_4\{24d(4, 1)$
$- 48d(4, 2) + 24d(4, 3)\}.$

$\underline{k = 6}$ $\gamma_6 = N^{(N+6)/2}\pi^{-N}$, $\Delta_6 = N^{(N+6)/2}2^{-2N}/6!$

$$D_{IN}(6; 6, 0, \cdots, 0) = \gamma_6\{32b(13, 1) - 40b(11, 1) + 12b(9, 1) - \tfrac{1}{2}b(7, 1)\};$$
$$\Delta_6\{d(6, 1) - \tfrac{63}{2}d(6, 2) + 105d(6, 3) - \tfrac{315}{4}d(6, 4)\}.$$

$$D_{IN}(6; 5, 1, 0, \cdots, 0) = \gamma_6\{32b(13, 1) - 32b(11, 1) + 6b(9, 1)\};$$
$$\Delta_6\{6d(6, 1) - 96d(6, 2) + 225d(6, 3) - 135d(6, 4)\}.$$

$$D_{IN}(6; 4, 2, 0, \cdots, 0) = \gamma_6\{\tfrac{15}{2}b(7, 1) - 40b(7, 3) + 64b(7, 5) - 32b(7, 7)\};$$
$$\Delta_6\{15d(6, 1) - 135d(6, 2) + \tfrac{1125}{4}d(6, 3) - \tfrac{675}{4}d(6, 4)\}.$$

$$D_{IN}(6; 4, 1, 1, 0, \cdots, 0) = \gamma_6\{10b(9, 1) - 40b(9, 3) + 32b(9, 5)\};$$
$$\Delta_6\{30d(6, 1) - 255d(6, 2) + 450d(6, 3) - 225d(6, 4)\}.$$

$$D_{IN}(6; 3, 3, 0, \cdots, 0) = \gamma_6\{32b(13, 1) - 32b(11, 1) + 8b(9, 1)\};$$
$$\Delta_6\{20d(6, 1) - 140d(6, 2) + 300d(6, 3) - 180d(6, 4)\}.$$

$$D_{IN}(6; 3, 2, 1, 0, \cdots, 0) = \gamma_6\{24b(11, 1) - 12b(9, 1) - 32b(11, 3) + 16b(9, 3)\};$$
$$\Delta_6\{60d(6, 1) - 330d(6, 2) + 540d(6, 3) - 270d(6, 4)\}.$$

$$D_{IN}(6; 3, 1, 1, 1, 0, \cdots, 0) = \gamma_6\{32b(13, 1) - 16b(11, 1)\};$$
$$\Delta_6\{120d(6, 1) - 600d(6, 2) + 840d(6, 3) - 360d(6, 4)\}.$$

$$D_{IN}(6; 2, 2, 2, 0, \cdots, 0) = \gamma_6\{\tfrac{27}{2}b(7, 1) - 54b(7, 3) + 72b(7, 5) - 32b(7, 7)\};$$
$$\Delta_6\{90d(6, 1) - 405d(6, 2) + \tfrac{1215}{2}d(6, 3) - \tfrac{1215}{4}d(6, 4)\}.$$

$$D_{IN}(6; 2, 2, 1, 1, 0, \cdots, 0) = \gamma_6\{18b(9,1) - 48b(9,3) + 32b(9,5)\};$$
$$\Delta_6\{180d(6,1) - 720d(6,2) + 945d(6,3) - 405d(6,4)\}.$$

$$D_{IN}(6; 2, 1, 1, 1, 1, 0, \cdots, 0) = \gamma_6\{24b(11,1) - 32b(11,3)\};$$
$$\Delta_6\{360d(6,1) - 1260d(6,2) + 1440d(6,3) - 540d(6,4)\}.$$

$$D_{IN}(6; 1, 1, 1, 1, 1, 1, 0, \cdots, 0) = 32\gamma_6 b(13,1);$$
$$\Delta_6\{720d(6,1) - 2160d(6,2) + 2160d(6,3) - 720d(6,4)\}.$$

The distribution functions of t may now be obtained. The sets of coefficients (6.9 CS) are combined with the set (5.3) in (4.13). For the Cauchy density,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d^s B(\Phi)}{d\Phi^s}\bigg|_{\Phi=0} P(z) \prod_1^n dz_i$$
$$= \frac{S_{n-1}}{2\pi^N} N^{(n-1)/2} \beta\left(\frac{n+1}{2}, \frac{n+1}{2}\right) \omega_s,$$

(6.10 C) $$s = 0, 2, 4, 6, \qquad \omega_0 = 1, \qquad \omega_2 = \frac{n+1}{n+2},$$

$$\omega_4 = \frac{3(n+1)(-2n^3 - 5n^2 + 29n + 14)}{(n+2)^2(n+4)}$$

$$\omega_6 = \frac{15(n+1)(-14n^5 - 375n^4 - 146n^3 + 3927n^2 + 3274n + 48)}{(n+2)^2(n+4)^2(n+6)}.$$

In the second example, for the squared hyperbolic secant density,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d^s B(\Phi)}{d\Phi^s}\bigg|_{\Phi=0} P(z) \prod_1^n dz_i = \frac{S_{n-1}}{2^{2n}} N^{(n-1)/2} \lambda_s,$$

$$s = 0, 2, 4, 6, \qquad \lambda_0 = I(n, n+1),$$

$$\lambda_2 = \frac{-(n+1)}{2} I(n+2, n+2),$$

$$\text{(6.11 S)}\quad \lambda_4 = \frac{3(n+1)}{2^2(n+2)}\{(n^2+6n+2)I(n+4,n+3) - 2nI(n+4,n+2)\},$$

$$\lambda_6 = \frac{-15(n+1)}{2^3(n+2)(n+4)}\{4n^2I(n+6,n+2) - (6n^3+68n^2+24n-8)I(n+6,n+3) + (n^4+17n^3+104n^2-230n+288)I(n+6,n+4)\}.$$

The cumulative distribution function of t for the Cauchy distribution is at once found by substituting the values of (6.10 C) in (3.11). S_{n-1} is replaced by its value (5.1). When $t > 0$,

$$\text{(6.12 C)}\quad G_N(t) = 1 - \frac{N^{N/2}n^{(n-2)/2}}{\pi^{(N+1)/2}}\frac{\beta\left(\frac{n+1}{2},\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)t^n}\cdot\left\{1 + \frac{n^2(n+1)}{2(n+2)^2}\left(\frac{1}{t^2}\right) + \frac{n^3(n+1)(14+29n-5n^2-2n^3)}{2^2 2!(n+2)^2(n+4)^2}\cdot\left(\frac{1}{t^4}\right) + \frac{n^4(n+1)(48+3274n+3927n^2-1460n^3-375n^4-14n^5)}{2^3 3!(n+2)^2(n+4)^2(n+6)^2}\cdot\left(\frac{1}{t^6}\right) + 0(t^{-8})\right\}.$$

Similarly, for the squared hyperbolic secant density,

$$\text{(6.13 S)}\quad G_N(t) = 1 - \frac{2\pi^{n/2}N^{(N+2)/2}n^{(n-2)/2}}{2^{2N}\Gamma(n/2)t^n}\cdot\left[\frac{I(n,n+1)}{(n+1)} - \frac{n^2I(n+2,n+2)}{2.2!(n+2)}\left(\frac{1}{t^2}\right) + \frac{3n^3\{(n^2+6n+2)I(n+4,n+3) - 2nI(n+4,n+2)\}}{2^2 4!(n+2)(n+4)}\cdot\left(\frac{1}{t^4}\right) - \frac{15n^4}{2^3 6!(n+2)(n+4)(n+6)}\{4n^2I(n+6,n+2) - (6n^3+68n^2+24n-8)\cdot I(n+6,n+3) + (n^4+17n^3+104n^2-230n+288)\cdot I(n+6,n+4)\}\left(\frac{1}{t^6}\right) + 0(t^{-8})\right].$$

$G_N(t)$ has been evaluated for sample sizes up to five. For the Cauchy density, we have the series,

$$
\begin{aligned}
G_2(t) &= 1 - \frac{2}{\pi^2 t}\left\{1 + \frac{1}{3^2 t^2} + \frac{1}{5^2 t^4} + \frac{1}{7^2 t^6} + 0(t^{-8})\right\},\\
G_3(t) &= 1 - \frac{3\sqrt{3}}{\pi 2^3 t^2}\left\{1 + \frac{3}{8t^2} + \frac{3}{16t^4} + \frac{29}{256t^6} + 0(t^{-8})\right\},\\
G_4(t) &= 1 - \frac{16}{\sqrt{3}\pi^3 t^3}\left\{1 + \frac{18}{5^2 t^2} + \frac{27}{5^2 7^2 t^4} - \frac{2332}{5^2 7^2 t^6} + 0(t^{-8})\right\},\\
G_5(t) &= 1 - \frac{75\sqrt{5}}{32\pi^2 t^4}\left\{1 + \frac{10}{9t^2} - \frac{65}{48t^4} - \frac{355}{24t^6} + 0(t^{-8})\right\};
\end{aligned}
\tag{6.14 C}
$$

and for the square hyperbolic secant density, we have

$$
\begin{aligned}
G_2(t) &= 1 - \frac{0.29543}{t}\left\{1 - \frac{0.11983}{t^2} + \frac{0.01686}{t^4} + \frac{0.00233}{t^6} + 0(t^{-8})\right\},\\
G_3(t) &= 1 - \frac{0.43342}{t^2}\left\{1 - \frac{0.76466}{t^2} + \frac{0.87123}{t^4} + \frac{1.17213}{t^6} + 0(t^{-8})\right\},\\
G_4(t) &= 1 - \frac{0.55910}{t^3}\left\{1 - \frac{4.01820}{t^2} + \frac{2.14852}{t^4} + \frac{3.39309}{t^6} + 0(t^{-8})\right\},\\
G_5(t) &= 1 - \frac{1.81014}{t^4}\left\{1 - \frac{12.49756}{t^2} - \frac{1.61817}{t^4} + 0(t^{-6})\right\}.
\end{aligned}
\tag{6.15 S}
$$

The values of $I(p, q)$ in Table I have been substituted in (6.13 S) to obtain this second set of distribution functions.

The probability that $|t|$ exceeds some t_0 may be obtained directly from $G_N(t)$ for both examples since each has a symmetric density function.[7] Figures I and II show the probability of $|t|$ exceeding t_0 plotted against t_0 for the Cauchy examples of (6.14 C). A logarithmic scale is used for t_0 and values for the "Student" distribution are also shown for the purpose of comparison. Figures III and IV show similar plots for the squared hyperbolic secant example.

[7] For populations with skew density functions it is necessary to use (3.13) as well as (3.11).

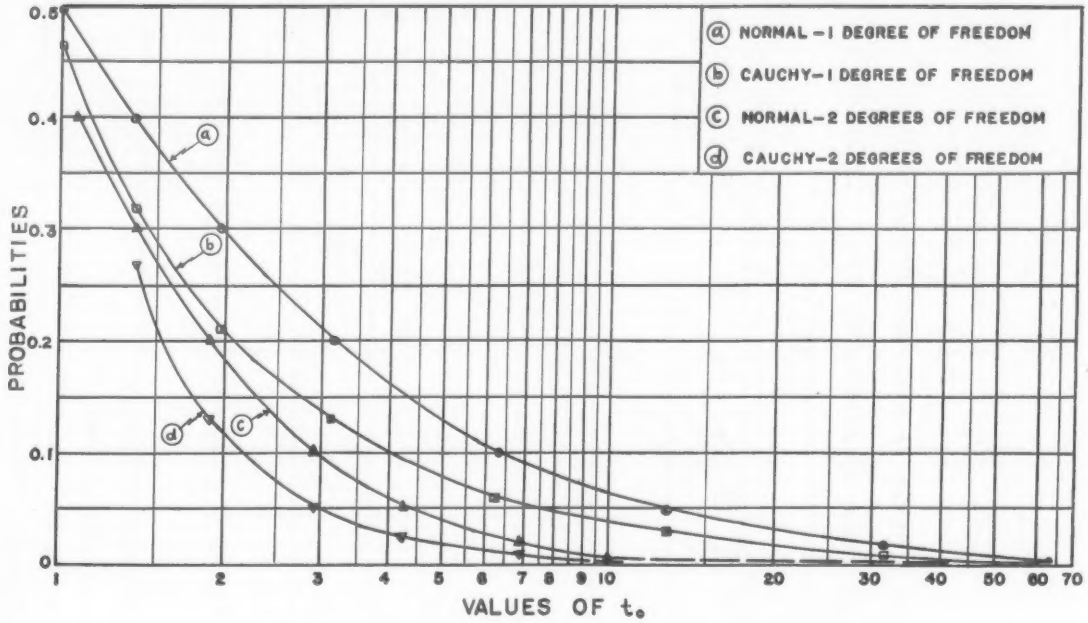

FIGURE I. THE PROBABILITIES THAT $|t| \geq t_o$ FOR THE NORMAL AND CAUCHY POPULATIONS

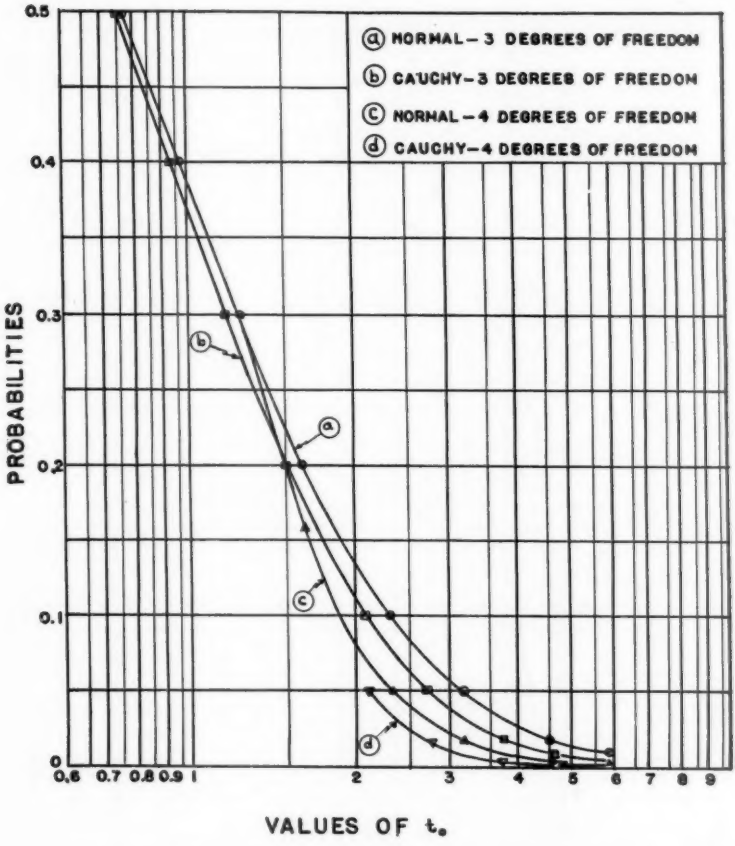

FIGURE II. THE PROBABILITIES THAT $|t| \geq t_o$ FOR THE NORMAL AND CAUCHY POPULATIONS

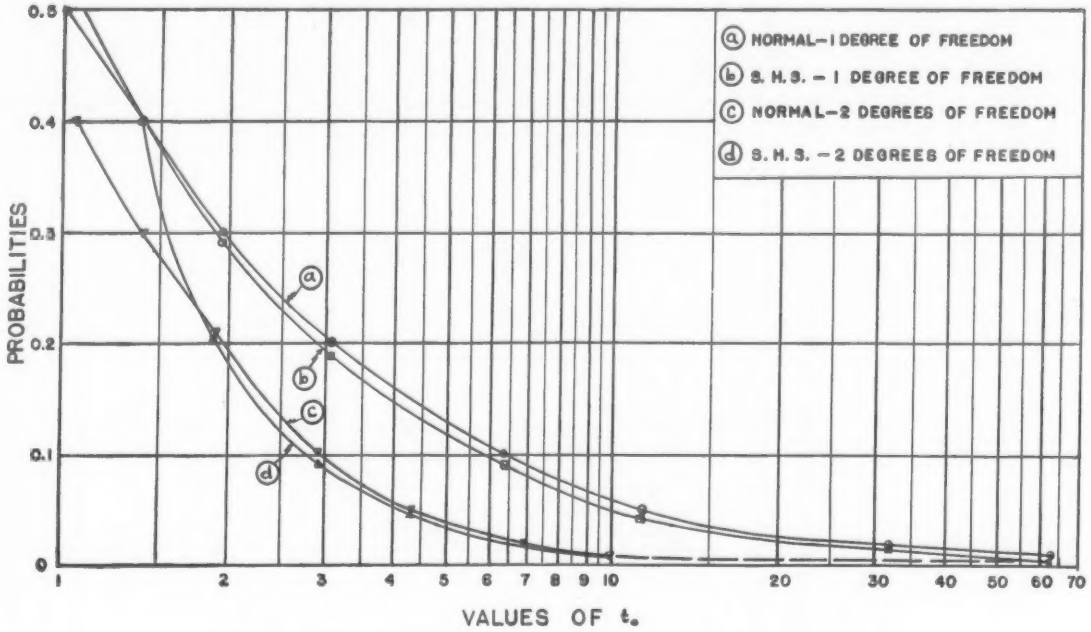
ⓐ NORMAL—1 DEGREE OF FREEDOM
ⓑ S. H. S. — 1 DEGREE OF FREEDOM
ⓒ NORMAL—2 DEGREES OF FREEDOM
ⓓ S. H. S. — 2 DEGREES OF FREEDOM
PROBABILITIES
0.5
0.4
0.3
0.2
0.1
0
1 2 3 4 5 6 7 8 9 10 20 30 40 50 60 70
VALUES OF t_o

FIGURE III. THE PROBABILITIES THAT $|t| \geq t_o$ FOR THE NORMAL AND SQUARED HYPERBOLIC SECANT POPULATIONS

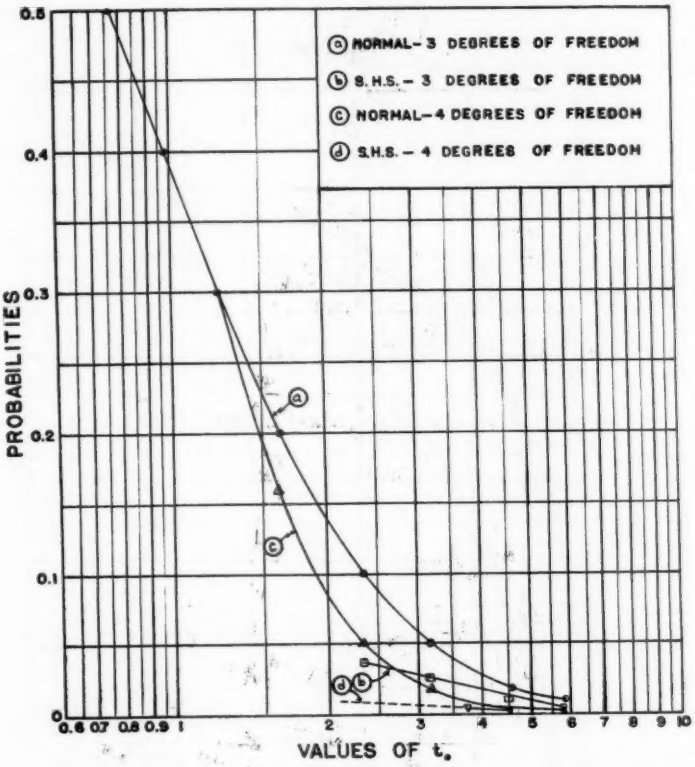

FIGURE IV. THE PROBABILITIES THAT $|t| \geq t_0$ FOR THE NORMAL AND SQUARED HYPERBOLIC SECANT POPULATIONS

THE GENERALIZATION TO TWO OR MORE SAMPLES

We have obtained general expressions for the density function and the cumulative distribution function of t for a sample of size N from populations whose density functions $f(u)$, satisfy certain specified conditions. The methods employed may be generalized to include the analogous two-sample and k-sample problems when the populations are similarly described. The sequences of transformations which have been used permit the required generalizations and, when more than one sample is considered, the linear and spherical transformations are performed in the same order as before. We state the results obtained and again refer the reader to footnote 2.

In this section, Σ denotes a summation over i, with i taking the values $1, \cdots, k$ and S denotes a summation over α for the values $1, \cdots, N_i$. Π and M represent respectively the corresponding products and Σ', S', Π' and M' are sum and product symbols defined as above with the exception that their upper limits of operation are all reduced by unity. N_i is the number of observations in the i^{th} sample.

The t statistic used to test hypotheses on the means of two samples, $x_{1i}, \cdots, x_{N_i i}$, $i = 1, 2$, when the samples are assumed to come from populations whose density functions differ at most in location, is

$$t = N^{-1/2}(nN_1N_2)^{1/2}(\bar{x}_1 - \bar{x}_2)/\{\Sigma S(x_{\alpha i} - \bar{x}_i)^2\}^{1/2}, \tag{7.1}$$

where $N = N_1 + N_2$, $n = N - 2$, and Σ is the described summation with $k = 2$. On the hypothesis that the means of the populations from which the samples are drawn are equal, each population may be described by the density function, $f(u)$. Then, the density function of t is

$$\begin{aligned} G'(t) = N(N_1N_2)^{1/2}n^{n/2}t^{-(n+1)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \\ \cdot[\Pi M' f\{u - (-1)^i(N_1/N_2)^{(-1)^i/2}P(u) + \sqrt{n}\, t^{-1}u_{\alpha i}\}] \\ \cdot[\Pi f\{u - (-1)^i(N_1/N_2)^{(-1)^i/2}P(u) - \sqrt{n}\, t^{-1}S'u_{\alpha i}\}] \\ P(u)\, du\, \Pi M'\, du_{\alpha i} \end{aligned} \tag{7.2}$$

with

$$P(u) = N^{-1/2}\{\Sigma S'u_{\alpha i}^2 + \Sigma(S'u_{\alpha i})^2\}^{1/2}.$$

The cumulative distribution function of t may be expanded in a series as before. This expansion is

$$G(t) = 1 - N(N_1N_2)^{1/2}\Gamma(n)n^{n/2}\left\{\sum_{s=0}^{\infty}\frac{n^s}{(n+2s)!}\right.$$

(7.3)
$$\left.\cdot\binom{n+2s-1}{2s}t^{-(n+2s)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{d^{2s}B(\Phi)}{d\Phi^{2s}}\bigg|_{\Phi=0}P(u)\,du\Pi M'\,du_{ai}\right\}$$

when $t > 0$ and provided the series converges. Terms in the series may be simplified since

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{d^{2s}B(\Phi)}{d\Phi^{2s}}\bigg|_{\Phi=0}P(u)\,du\Pi M'\,du_{ai} = \tfrac{1}{2}(NN_1N_2)^{-1/2}(2s)!$$

(7.4)
$$\sum_{r=0}^{s}\beta\left(\frac{N_1+2r-1}{2},\frac{N_2+2s-2r-1}{2}\right)\sum_{\substack{\text{comb.}\\ a_0+\cdots+a_{N_1-1}=2r}}$$

$$\sum_{\substack{\text{comb.}\\ b_0+\cdots+b_{N_2-1}=2s-2r}}C_{IN_1}(2r;a)C_{IN_2}(2s-2r;b)D_{IN_1N_2}(2r,2s-2r;a,b)$$

with

$$D_{IN_1N_2}(2r,2s-2r;a,b) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\rho^{N+2s-2}$$

(7.5)
$$\left[\prod_{\alpha}\frac{f^{(a_\alpha-1)}\{u+(N_2/NN_1)^{1/2}\rho\}}{a_{\alpha-1}!}\right]$$

$$\left[\prod_{\beta}\frac{f^{(b_\beta-1)}\{u-(N_1/NN_2)^{1/2}\rho\}}{b_{\beta-1}!}\right]du\,\partial\rho.$$

The coefficients C_{IN_1} and C_{IN_2} are identical with those in (4.12) when N is replaced by N_1 and N_2. When $t < 0$, a similar series is obtained. Then, we define $D^*_{IN_1N_2}$ like $D_{IN_1N_2}$ with the exception that $-\rho$ replaces ρ in the arguments of f, and the cumulative distribution function of t becomes $1 - G^*(|t|)$ where $G^*(|t|)$ is $G(t)$ defined by (7.3) and (7.4) with $|t|$ replacing t and $D^*_{IN_1N_2}$ replacing $D_{IN_2N_2}$. The integrals $D_{IN_1N_2}$ and $D^*_{IN_1N_2}$ are of multiplicity two and must be evaluated for each density, $f(u)$, separately.

The analogous k-sample test depends on the statistic

(7.6)
$$F = n_2\Sigma N_i(\bar{x}_i - \bar{x})^2/\{n_1\Sigma S(x_{ai} - \bar{x}_i)^2\}$$

where $N = \Sigma N_i$, $n_1 = k - 1$, and $n_2 = N - k$. The density function of F may be shown to be

$$G'(F) = \tfrac{1}{2}N(N-N_k)N_{k-1}^{-1/2}\Pi N_i^{\frac{1}{2}}F^{-(\frac{1}{2}n_2+1)}$$

(7.7)
$$\sum_{\gamma=1}^{2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\Pi B_i^{\gamma})P(u)Q^{n_2}(u)\,du\prod_{1}^{k-2}du_i\prod M'\,du_{ai}$$

where $P(u) = \{\Sigma S' u_{ai}^2 + \Sigma (S' u_{ai})^2\}^{1/2} / \{N(N - N_k)\}^{1/2}$,

$$Q(u) = (n_2/n_1)^{1/2} \left\{ \Sigma S' u_{ai}^2 + \Sigma (S' u_{ai})^2 + \sum_1^{k-2} u_i^2 + N_{k-1}^{-1} \left(\sum_1^{k-2} \sqrt{N_i}\, u_i \right)^2 \right\}^{1/2} \{\Sigma S' u_{ai}^2 + \Sigma (S' u_{ai})^2\}^{-1/2},$$

$$B_i^\gamma = f\{U_i + (-1)^\gamma \lambda_i P(u) - F^{-1/2} Q(u) S' u_{ai}\} \qquad M' f\{U_i + (-1)^\gamma \lambda_i P(u) + F^{-1/2} Q(u) u_{ai}\}$$

with

$$U_i = u + N_i^{-1/2} u_i, \qquad i = 1, \cdots, (k-2),$$

$$U_{k-1} = u - N_{k-1}^{-1} \sum_1^{k-2} N_i^{1/2} u_i, \qquad U_k = u,$$

$$\lambda_i = N_k^{1/2}, \qquad i = 1, \cdots, (k-1), \qquad \lambda_k = -N_k^{-1/2}(N - N_k).$$

The cumulative distribution function of F becomes

$$G(F) = 1 - \Gamma(n_2) F^{-n_2/2} \left\{ \sum_{s=0}^{\infty} \frac{F^{-s}}{(n_2 + 2s)!} \binom{n_2 + 2s - 1}{2s} (n_2/n_1)^s \right.$$

(7.8)
$$\left. \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{\gamma=1}^{2} \frac{d^{2s} B^\gamma(\Phi)}{d\Phi^{2s}} \Big|_{\Phi=0} P(u) Q^{n_2}(u)\, du \prod_1^{k-2} du_i \Pi M'\, du_{ai} \right\}.$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{\gamma=1}^{2} \frac{d^{2s} B^\gamma(\Phi)}{d\Phi^{2s}} \Big|_{\Phi=0} P(u) Q^{n_2}(u)\, du \prod_1^{k-2} du_i \Pi M' du_{ai}$$

(7.9)
$$= \frac{2^{-n_1}(2s)!\, \Pi N_i^{1/2}}{\Gamma\left(\frac{n_2}{2} + s\right)} (n_2/n_1)^{n_2/2} \sum_{(d)}^{p} \left[\sum_{(a_1)}^{c} \cdots \sum_{(a_k)}^{c} \right.$$

$$\left. \left\{ \Pi \Gamma\left(\frac{N_i + 2d_i - 1}{2} \right) C_{IN_i}(2d_i; a_i) \right\} D_I(2d_1, \cdots, 2d_k; a_1, \cdots, a_k) \right]$$

where $\sum_{(d)}^{p}$ denotes a summation over all permutations $d_1 + \cdots + d_k = s$, $\sum_{(a_i)}^{c}$ is a summation over all combinations $a_{0i} + \cdots + a_{N_i - 1, i} = 2d_i$, $i = 1, \cdots, k$, and C_{IN_i} is the same set of coefficients obtained before. The explanation of (7.9) is complete when we write

$$D_I(2d_1, \cdots, 2d_k; a_1, \cdots, a_k)$$

(7.10)
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{\Sigma N_i w_i^2 - N^{-1}(\Sigma N_i w_i)^2\}^{\frac{1}{2}n_2 + s} \prod \prod_{\alpha=0}^{N_i - 1} \frac{f^{(a_{\alpha i})}(w_i)}{a_{\alpha i}!} \Pi\, dw_i.$$

D_I is a k-fold integral which must be evaluated for each population considered. It depends indirectly on d_i and s since $\Sigma d_i = s$ and $\sum_{\alpha=0}^{N_i-1} a_{\alpha i} = 2d_i$; a_i represents the vector, $(a_{0i}, \cdots, a_{N_i-1,i})$.

It may be shown that when $n_1 = 1$, the expression (7.8) for $G(F)$ is equivalent to the cumulative distribution function of t^2 obtained by combining $G(t)$ in (7.3) with the similar expression for the cumulative distribution function of t when $t < 0$.

DISCUSSION

The examples on the Cauchy and the squared hyperbolic secant populations indicate that as the sample size increases, more terms of the series may be required to calculate the probabilities that specified values of the statistics be exceeded. The rapidity of convergence appears to decrease as the sample size increases and the value of the statistic decreases. Due to the complexity of the methods, a consideration of the convergence of the distribution functions was unsuccessful. It is hoped that further work may be forthcoming on this point.

It is interesting to note that the coefficients, C_{IN}, appear in all three problems and need only to be evaluated once since they are independent of the population considered. These coefficients have been evaluated for the first four terms of the series but become increasingly numerous and complex as more terms are considered. Additional values should be obtained but require improved computing methods.

The series expansions may be used to obtain the power functions of the statistics t and F. $f(u)$ may be replaced by $f(u - a)$, where a is a measure of location, and the coefficients C_{IN} will be unchanged. When both scale and location parameters are present, the measure of noncentrality will be a/c, where c is the scale parameter. Some difficulty may be introduced in the evaluation of the D-integrals but their multiplicities will be unchanged.

ACKNOWLEDGMENT

The content of this paper is part of a dissertation submitted to the Department of Mathematical Statistics of the University of North Carolina in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The writer is indebted to Professor Harold Hotelling for guidance and encouragement in this work.

The manuscript was prepared using the facilities of the Department of Statistics and the Statistical Laboratory of the Virginia Polytechnic

Institute and the Virginia Agricultural Experiment Station. The figures were drawn by Mr. G. J. Lombardi.

SUMMARY

The probability density functions of t and of F have been expressed as multiple integrals of multiplicity one less than the sample size and whose ranges of integration are unlimited. The methods are appropriate for non-normal density functions satisfying rather mild conditions.

The Taylor expansion is used to expand the cumulative distribution functions of t and of F in power series in t^{-1} and F^{-1}. The series are based on the density functions described immediately above. The coefficients in the expansion depend on integrals whose multiplicities are equal to the number of groups of observations in the sample. Two examples are included in section 6. They substantiate the opinion that the effects of moderate departures from normality may not be serious in the analysis of variance.

Certain problems associated with this work still require solution. However, this is the first study by analytical methods of the effects of non-normality for a general class of populations using specified density functions.

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A Comparison of the Gross Effects of Insulin Injection in Normal Mammals, Birds and Poikilotherms

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In the period which has elapsed since Banting and Best (1922) first demonstrated the internal secretion of a hormone, insulin, by the islets of Langerhans in the pancreas, a vast amount of research has been directed at elucidating the various functions performed by the hormone in the living organism. Much of this research has necessarily involved the use of animals or specific tissues of animals of a variety of species. The results obtained have been frequently applied *a priori* to the human species. That different species react differently to removal of the pancreas has been pointed out by Peters and Van Slyke (1946). It is the purpose of the present communication to review the effects produced by administration of insulin to intact animals of different species.

The concentration of insulin in the pancreas is reasonably constant from species to species. Normal values have been reported for fish, chickens, ducks, mice, rats, guinea pigs, rabbits, cats, dogs, cattle, pigs, sheep, horses, monkeys, chimpanzees and men (Haist, 1944). The insulin preparation used almost exclusively at the present time is isolated from beef and pork pancreas. While this hormone preparation has qualitatively similar effects in all vertebrates, it has been shown that there are gross quantitative differences in the reactions of various species to injection of insulin (Macleod, 1923, 1926).

Effects in mammals.—The main effects of injecting insulin into mammals are very striking. The blood glucose level falls rapidly and the animal is said to be in a state of hypoglycemia. If the level of blood glucose falls below a definite value, which differs from one mammal to another, highly characteristic symptoms supervene. These are known as the "hypoglycemic symptoms" and, in general, are characterized by convulsions or coma, or both.

In the rabbit, the fall in blood glucose sets in almost immediately after the injection of insulin so that after thirty to forty-five minutes it stands at about one-half its normal value. Definite convulsions appear if the blood glucose level falls below this value (45 mg. %) which usually occurs following a dose of one to five units[1] per kg. body weight. The

1. By definition 1 mg. of zinc insulin crystals contains 22 international units of insulin.

typical effect of insulin on the blood glucose of the dog is very similar to that observed in rabbits, with the difference being that the fall is not as steep. The minimum blood glucose level (about one-half the normal level) is reached in two hours or more, and when the dose is about 23 units per kg., convulsions are observed in from four to five hours. The cat and rat react in much the same manner as the dog. In man the blood glucose decreases more slowly after insulin injection. When the concentration of blood drops to 70 or 60 mg. %, the general symptoms may become so severe that it is necessary to administer glucose as an antidote. However, in man, variations between individuals are very great in this respect. A blood glucose level of 35 mg. % is usually accompanied by unconsciousness. Among the mammals on which good information is available, the mouse, rat, dog, and man display about equal sensitivity to insulin while the rabbit is three to four times as susceptible.

Effects in birds.—It has been established that birds, as a whole, are affected less by insulin than mammals (Chen *et al.*, 1945, Sordelli *et al.*, 1923). However, even among different species of birds, there is a wide difference in sensitivity. Both the horned owl (*Bubo virginianus*) (Nelson *et al.*, 1942) and the Pekin duck (Chen *et al.*, 1945) are relatively sensitive to insulin, the latter having convulsions as early as two hours after receiving adequate doses of insulin. The pigeon is more tolerant to insulin than the duck, and some specimens can withstand up to 150 units within 24 hours, showing only a 50% reduction in blood glucose (Stare and Baumann, 1940). Domestic fowl (white Leghorns, Plymouth rocks, Rhode Island reds), however, appear to be very resistant to insulin; doses of 10 to 130 units per kg. may lower the blood glucose level to 25% of the normal, but although the birds get weak when the blood glucose is so low, no convulsions appear (Cassidy *et al.*, 1926). The observations of Chen and his co-workers (1945) on roosters (white Leghorn) are in accord with this finding. They report that, of nine roosters injected with doses varying from 100 to 500 units of insulin per kg., only one had ataxia and two fell on their sides, but all appeared normal the next day.

Effects in poikilotherms.—Cold-blooded vertebrates also exhibit a marked resistance to insulin. Moreover, it was noted very early that in cold-blooded animals it was impossible to correlate the symptoms usually associated with hypoglycemia with the blood glucose level. In some species the blood glucose level may fall, but convulsions do not appear until much later, while in other species insulin injection may cause convulsions but the fall in the blood glucose, if it occurs at all, is not detected for a matter of days later.

Houssay and his co-workers (1924, Sordelli *et al.*, 1923) found that injection of large amounts of crude insulin into a variety of cold-blooded animals of the Argentine (including some reptiles) eventually

lowered the blood glucose level and produced convulsions. But the appearance of the two effects did not coincide in these animals, and all animals died several days after the injections. Small crocodiles (*Caiman sclerops*), for example, were hypoglycemic one day after crude insulin injections, showed hypoglycemic symptoms on the third or fourth day, and died in seven to ten days (Houssay and Rietti, 1924).

The cold-blooded animal which has been most thoroughly studied in relation to insulin effects has been the frog. Abderhalden (1924) was one of the first to note that frogs tolerate remarkably high doses of insulin. In more detailed studies, Olmsted (1924, 1927) and Huxley and Fulton (1924) observed that frogs at room temperature did not convulse earlier than 24 hours after insulin injection. The insulin convulsions in frogs have been described accurately, and it has been observed that the onset of the symptoms is greatly accelerated by warming the injected frogs (Huxley and Fulton, 1924, Olmsted, 1924). The same result obtains when insulin-injected toads are warmed (Olmsted, 1926). More detailed studies of the effects of insulin on the blood glucose level and the onset of convulsions were made by Gabbe (1925) with the European frog (*Rana temporaria*) and by Barlow, Vigor and Peck (1931) with "grass frogs" (probably *Rana pipiens*). In both species it was found that the appearance of convulsions anteceded, and was apparently independent of, the drop in blood glucose concentration. Gourley and Fisher (1951) confirmed these observations in *Rana pipiens*, and found that the injection of 3.0×10^{-3} units of insulin per ml. of body fluid caused convulsions to occur within 24 hours and a 50% decrease in the blood glucose concentration after 48 hours. A concentration of 10^{-4} units of insulin per ml. of body fluid had none of these effects. Olmsted (1924), who thought that the blood glucose level of insulin-injected frogs remained normal, failed to observe hypoglycemia only because he made his determinations at the time of convulsions. It will be noted that the order of appearance of the two effects in mammals is reversed in the frog.

Olmsted (1924) described peculiar symptoms, considered by him to be related to hypoglycemia, which were observed two days after the injection of insulin into fresh water catfish (*Ameiurus nebulosus*). McCormick and Macleod (1925) observed similar symptoms in the sculpin (*Myoxocephalus*) and the sea raven (*Hemitripterus*) several days after insulin injection but were unable to demonstrate that the insulin injection caused the normally low blood glucose to become further lowered. Macleod (1926) was inclined to feel that asphyxia rather than hypoglycemia may have been partly responsible for the symptoms observed. The possibility of asphyxia was eliminated in the carefully controlled investigation of Gray and Hall (1930) which indicated that the very active fishes, menhaden (*Brevoortia tyrannus*), common mackerel (*Scomber scombrus*) and bull's eye mackerel (*Pneumatophorus colias*), suffered insulin shock in a much shorter time than did the moderately active scup (*Stenotomus*

chrysops). The more sluggish bottom feeders, sea robin (*Prionotus carolinus*), puffer (*Spheroides maculatus*) and toadfish (*Opsanus tau*), showed no external evidences of the effects of the hormone. Insulin appeared to reduce the blood glucose concentration of fishes in much the same manner as in mammals, except that a longer time was required for the effect to occur. It is curious that in fishes, as in birds, the blood glucose may be lowered without any signs of convulsions appearing. This fact led Gray and Hall to conclude that insulin shock in fishes does not have so much significance as has been attributed to insulin convulsions in mammals.

Summary and conclusions.—The association of the blood glucose level to the incidence of hypoglycemic symptoms in mammals suggests a causal relationship between the two (Himwich, 1944). The suggestion appears to be sound in view of the fact that the hypoglycemia which follows removal of the liver is also accompanied by the same symptoms. The symptoms are relieved immediately in both instances by the administration of glucose. The facts reviewed above, however, indicate that the same relationship does not hold in birds and in poikilotherms. Even in mammals there are many exceptions (Barlow *et al.*, 1931, p. 236; Gabbe, 1925, p. 215; Harris, 1944, p. 354; Peters and Van Slyke, 1946, p. 229).

It is clear then that distinctions between the reactions of different species to insulin exist. Mammals appear to be the most sensitive to insulin while birds are most resistant to the hormone. An extensive study of the effect of insulin in reptiles has not yet been made but an amphibian, the frog, appears to be somewhere between mammals and birds in insulin sensitivity. Peters and Van Slyke (1946) suggest that these differences may be related to differences in the inherent metabolic processes upon which the action of insulin is superimposed. These processes are the result of evolutionary adaptations and insulin does not bring them into existence but may, by influencing the rates of certain ones, alter the direction of the metabolism as a whole.

It is interesting to speculate on the possible reason for the species difference in reactions to insulin. The birds of the present period constitute an extraordinarily successful and highly diversified group, probably at the height of their evolutionary career (Newman, 1939). It seems conceivable, therefore, that birds might have developed an independence of whatever reactions are controlled by insulin. In mammals, on the other hand, certain processes might conceivably have become more dependent on the action of this hormone. A similar evolutionary change has been suggested by Beadle (1945) to account for the manner in which successive generations of microorganisms lose some of their synthetic abilities and become more exacting in their nutritional requirements. The ancestry of both birds and mammals has been traced back to primitive reptiles, which in turn developed from the primitive amphibians. As has already been noted, frogs are more sensitive to insulin than birds but

less sensitive than mammals. It is necessary to suppose in the present speculation that, in respect to its insulin sensitivity at least, the modern frog has not changed greatly from the primitive amphibians from which it descended.

Whatever reasons lie behind the species distinctions in the gross effects of insulin, it is clear that in animal experiments designed to elucidate the mechanisms of the hormone's action analogies to other species must be applied with caution.

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The Exoskeleton and Musculature of the Appendages of the First Three Abdominal Segments of *Cambarus Longulus Longulus* Girard (Decapoda, Astacidae)[1]

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INTRODUCTION

Since Schmidt (1915) extended the pioneer work of Huxley (1880) by rendering a complete account of the musculature of *Astacus fluviatilis* (Astacinae) no comparable study has appeared which deals with this or other groups of crayfishes. Since the Cambarinae is a group, restricted to the North American continent east of the Rocky Mountains (except for introductions by man into California, Japan and Western Europe) it seems desirable to examine the musculature of the appendage of this related but anatomically distinct subfamily as exhibited by *Cambarus longulus longulus* Girard.[2]

The anatomy of the first three abdominal segments, their appendages, and a study of the appendicular musculature of this crayfish has been found, except for minor details, to be identical with Schmidt's description of *Astacus fluviatilis*.

The muscular terminology employed by Schmidt is adopted in this study.

I wish to express my thanks to Dr. Horton H. Hobbs, Jr. who has read and criticized this manuscript.

THE EXOSKELETAL CONSTRUCTION OF THE ABDOMINAL SEGMENTS

Several rather distinct regions of the exoskeleton of the abdominal segments have been distinguished by previous investigators, and although these regions constitute a continuous whole, it is convenient to consider each segment to be composed of four basic sclerites—a *tergite*, a pair of *pleurites*, and a *sternite* (figures 1-8 and 10).

The sternum of the first abdominal segment is so constructed as to form a heavy transverse ridge which projects ventrally to a marked degree. It is on this ridge that the first pair of pleopods articulate

1. Contribution from the Miller School of Biology, University of Va.

2. The specimens on which this study is based were collected from Swift Run, a tributary of the James River, seven miles west of Stanardsville, Green Co., Va. on U.S. Hy. 33, and Keswick Creek, eight miles east of Charlottesville, Albemarle Co., Va. on State Hy. 22.

(figs. 1 and 10). This sternal region is clearly demarked from the pleural sclerites by obvious sutures (fig. 1), and the muscles which control the pleopods have their origins on the sternal sclerite. This condition is quite different from that found in the other segments of the abdomen where the sterno-epimeral juncture is at the mid-point of the articular cavities and the muscles have their origins on the tergum (figures 2-6). However, the serial homologies of these muscles are quite evident throughout the abdominal segments. In spite of the modifications of the first and second abdominal appendages, the first three show similarities in their structure.

In view of the fact that the third pleopod is a more generalized appendage than the first and second, it is used here as a basis for comparison of the three.

THE THIRD PLEOPOD

GROSS ANATOMY—This appendage, which is attached at the articular cavity situated between the sternal and the pleural sclerites of the third abdominal segment, consists of a stalk which is composed of a short basal joint, the *coxopodite,* followed by a long cylindrical second joint, the *basipodite* (figures 3 and 4). The coxopodite forms a semicircular plate, and this together with the basipodite is called the *protopodite.* At the distal end of the basipodite there are borne two narrow, flattened, multi-articulate rami, one of which is called the *endopodite,* and the other, which is fixed slightly more distad on the lateral side of the basipodite, the *exopodite.*

The endopodite, which is larger than the exopodite, is undivided for almost half its length. The multi-articular distal portion, the *flagellum,* is narrow and flattened and is divided into a number of small "rings" distinguished from one another by slight constrictions of the exoskeleton. The exopodite is a similar structure, the undivided proximal portion being shorter and not so wide. The edges of both the endopodite and the exopodite are fringed mesially, distally, and laterally with long setae.

MUSCULATURE (figures 2-4).—Schmidt (1915:238-241) names these muscles and gives an account of their function as he found them in *Astacus fluviatilis.* However, since his interpretations of the actions of a few of these do not seem to obtain in *Cambarus longulus longulus,* the muscles of this crayfish are described below and their probable functions indicated. The names of the muscles used in this paper are the same as those used by Schmidt.

Musculus remotor III. pedis spurii (rem.) has its origin in two heads situated somewhat caudad on the broad surface of the tergum. The two origins lie in a longitudinal axis, and the fibers, uniting proximally, are inserted caudally on the proximal margin of the coxopodite. It is this single remotor muscle which binds the coxopodite to the body and moves it backward in the longitudinal axis.

Musculus rotator dorsalis basipoditis III. pedis spurii (M. rot.d.) originates cephalad to the two heads of the remotor muscle on the broad surface of the tergum and is inserted on the proximal preaxial margin of the basipodite. It is subequal in size to that of the remotor muscle and moves the basipodite laterally.

Musculus rotator ventralis basipoditis III. pedis spurii (M.rot.v.) originates on the sternal sclerite immediately caudad to the articular cavity, and is inserted on the cephalic margin of the basipodite mesiad to the insertion of *M. rotator dorsalis*. It moves the basipodite mesially. A forward movement is obtained by a simultaneous contraction of the two rotator muscles. These two, together with the remotor, can move the basipodite in either a transverse or a longitudinal plane, and therefore are able to cause it to rotate.

Musculus reductor basipoditis III. pedis spurii (M.red.b.) originates caudally on the coxopodite distal to the insertion of the remotor, and extending through the entire length of the basipodite, is inserted postaxially on the distal end of the basipodite. The function of this muscle is dependent upon the relative positions of the coxopodite and the basipodite. This being one of the strongest muscles, it forms, together with the remotor, an antagonist for both of the rotators of the basipodite.

Musculus adductor endopoditis III. pedis spurii (M.add.end.) originates mesially near the proximal end of the basipodite and is inserted mesially on the proximal end of the endopodite.

Musculus abductor endopoditis III. pedis spurii (M.abd.end.) is weaker than the above mentioned muscles. A short muscle, it originates mesially in the distal third of the basipodite and terminates in a small tendon which is inserted on the proximolateral end of the endopodite.

Musculus abductor exopoditis III. pedis spurii. (M.abd.ex.) originates laterally on the proximal end of the basipodite, and is inserted laterally in the short basal part of the exopodite.

These last three muscles, *M. adductor endopoditis, M. abductor endopoditis,* and *M. abductor exopoditis* behave in such a manner that their actions are more easily understood if they are considered together. The proximal portions of the endopodite and the exopodite lie together so that the movement of one causes the movement of the other in the same direction. In this way the three muscles are able to serve as adductors and abductors of both the endopodite and the exopodite.

M. adductor endopoditis and *M. abductor endopoditis* serve a double purpose as adductors for the endopodite and abductors for the exopodite, while *M. abductor exopoditis* serves as an adductor for the exopodite and an abductor for the endopodite. *M. abductor endopoditis* is an abductor of the endopodite only in that it moves the endopodite laterally past the median axis toward the exopodite. *M. adductor endopoditis* also serves as an abductor of the endopodite in that it assists

M. abductor endopoditis. In the same manner ***M. abductor exopoditis*** is an abductor for the exopodite only if it moves the exopodite mesially past the median axis. The action of these muscles, just as in the case of ***M. reductor basipoditis,*** is dependent upon the relative positions of the joints involved.

Musculus flagellaris endopoditis III. pedis spurii (M. flag. end.) originates laterally in the proximal portion of the basal part of the endopodite just caudad to the insertion of ***M. adductor endopoditis.*** It traverses this segment to the distal end and gives off many sinews, each of which is inserted on a "ring" of the flagellum. Thus, by contractions, it is able to rotate the end of the endopodite.

Musculus flagellaris exopoditis III. pedis spurii (M. flag. ex.) originates laterally on the short basal segment of the exopodite and is inserted by sinews to the "rings" of the flagellum. The action of these two flagellar muscles is similar.

In the male crayfish, the third through the fifth pairs of pleopods are practically identical, and in the females the second pair of pleopods are unmodified and are similar to the third.

THE SECOND PLEOPOD

GROSS ANATOMY (figure 9).—The appendages of the first and second abdominal segments of the male crayfish are not only larger in size, but they are modified in other respects. The second pair are of the same general structure as the third and the basal joint (coxopodite) still retains its semi-circular structure. Here it is composed of two slightly overlapping sclerites instead of one continuous sclerite and they are designated as coxopodite "a" (cx.a.) and coxopodite "b" (cx.b.). The next joint, the basipodite, although larger, still retains the elongate cylindrical structure found in the third pair of appendages. Extending from the protopodite, just as in the third pair of appendages, are an exopodite and an endopodite, the former remaining similar to that of the third pleopod and the latter exhibiting a striking modification.

The basal half of the endopodite is cylindrical and is similar to the basipodite. On the mesial surface of the endopodite, and about three fourths its length from the base, is a subtriangular lamella, the apex of which is directed dorsally when the pleopod is held against the thorax. The distal fourth of the endopodite is a short segmented flagellum reminiscent of the longer multi-articulate portion of the endopodite of the third pleopod.

The exopodite of the second pleopod is comparable to that of the third except that it is of greater size.

MUSCULATURE.—In the second pair of pleopods there is little alteration in the musculature from that pattern found in the third pair, the principle difference occurring in the increase in size of the muscles that

pull the pleopod preaxially—particularly *M. rotator ventralis* (figs. 5, 6).

THE FIRST PLEOPOD

GROSS ANATOMY.—(fig. 12) The coxopodite, so easily seen in the other pleopods, has all but disappeared in the first. The two sclerites, ex.a. and ex.b., seen in the coxopodite of the second pleopod, are greatly reduced in size and no longer articulate with one another. Ex.a., (fig. 7) the sclerite on which the remotor muscle is inserted, lies at the caudolateral margin of the basipodite and is attached to it by a thick chitinous membrane.

The penultimate podomere, the basipodite, which is an obvious entity in the other pleopods, can be distinguished from the distal one only in the second form males.[3] Here the basipodite is delineated from the endopodite by a faintly perceptible line obliquely traversing the appendage (fig. 11). In the first form males this line can not be seen (fig. 12). The remaining two thirds of the pleopod, which exhibits the most bizarre modifications encountered thus far, is homologous to the endopodite of the other pleopods, as will be shown in a subsequent paper.

In the first pleopod there is no movable articulation between the basipodite and the endopodite, the whole appendage is rigid and the above mentioned oblique line serves only as a reminder that at some time in the evolution of the crayfish, here probably existed a movable joint.

MUSCULATURE.—(figs. 7, 8) It would seem that the first pair of pleopods of the male crayfish would behave like the others; however, their movement is largely confined to the longitudinal axis of the body. This rigid appendage is controlled by three strong muscles which correspond to *M. remotor*, *M. rotator dorsalis*, and *M. rotator ventralis* of the other pleopods. The remotor has two heads as does the corresponding muscle in the second and third pleopods. One head, originating laterally on the sternal sclerite, unites with the mesial head which arises from the caudal side of the sternal sclerite just laterad to the mid-point of the articular cavity and together they are inserted on the coxopodite. This muscle serves to pull the pleopod preaxially between the bases of the fifth pair of pereiopods.

The ventral rotator muscle has its origin on the caudal side of the first abdominal sternite and is inserted mesiad of the dorsal rotator muscle on the cephalic margin of the basipodite. The dorsal rotator muscle has its origin laterally on the sternal sclerite mesiad of the remotor and is inserted on the preaxial margin of the basipodite lateral to the insertion of the ventral rotator muscle. These two muscles are antagonistic to the remotor muscle and swing the pleopod postaxially. These three

3. "Second form males."—Those individuals which are in the non-breeding stage characterized by absence of corneous processes on the distal portion of the first pleopod and by blunt hooks on the ischiopodites. Immature males as well as alternating adult instars are typically second form.

muscles are the only ones found associated with the first pleopod and they are homologous to *M. remotor*, *M. rotator dorsalis*, and *M. rotator ventralis* found in the other pleopods.

THE VENTRAL SUPERFICIAL MUSCLES OF THE CAUDAL PORTION OF THE CEPHALOTHORAX AND THE FIRST THREE ABDOMINAL SEGMENTS

Figure 2

Musculus ventralis superficialis thoraco-abdominalis (v.s. th.a.A-B) extend from the thorax to the abdomen and fall into two groups, both of which originate mesially on the endophragmal skeletal apodemes of the last thoracic segment. The first branch, v.s.th.a.A, which is not shown in figure 2, is located ventrally to v.s.th.a.B and is inserted medio-ventrally on the interarticular membrane of the joint. The other muscle, v.s.th.a.B., is longer and more powerful and is inserted on the caudomesial portion of the sternum of the first abdominal segment.

Musculi ventrales superficiales abdominis (v.s.a.1-2-3-4) (Milne Edwards: La couche superficielle des muscles flechisseurs). The first pair of these muscles (v.s.a.1 and 2) originates on the interarticular membrane between the first and second abdominal sternites. Their origins are mediocaudad of the pleura of the first segment and they are inserted laterally near the caudal margin of the sternum of the second segment. The next pair of muscles (v.s.a.3 and 4) have their origins on the interarticular membrane between the second and third abdominal sternites just caudolaterad of each side of the sternite of the second segment and they are inserted laterally near the caudal margin of the sternite of the third segment.

These muscles are serially homologous and are present between all of the abdominal sternites. They are attached below the axes of the joints and give rise to flexion of the abdomen as a whole by approximating the sterna.

SUMMARY - CONCLUSIONS

The segmented abdominal exoskeleton of the crayfish consists of four sclerites—a tergite, a pair of pleurites, and a sternite—which are not always easily distinguished. The ventrolateral plates which project below the plane of the sternum may be considered to be tegal folds, reminiscent of the branchiostegites which form the lateral portions of the cephalothorax.

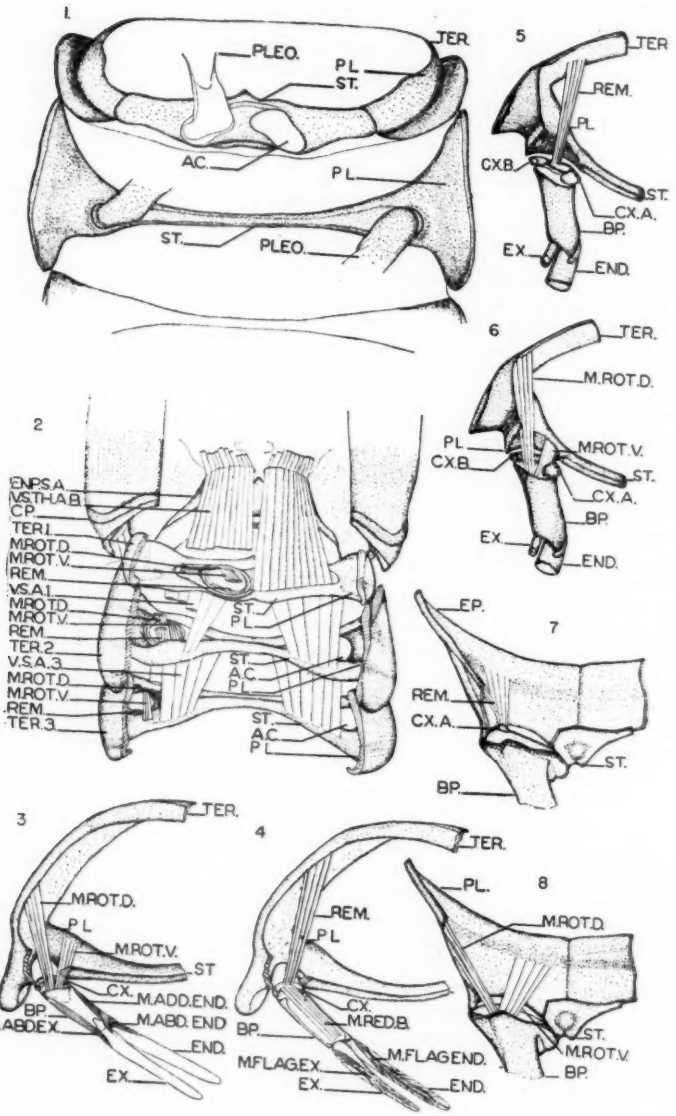
The elements which make up the ventral portion of the first abdominal segment are not easily homologized with those of the other segments for not only are the appendages of this segment highly modified, but also the sternal area on which they are borne is markedly different. It is considered here that the heavy ridge which bears the appendages is

homologous with the sternite of the other segments. Here the sternite has become longer (in the horizontal axis) with the articular cavities not being comparably displaced, thus leaving a considerably broader portion of the sternite lateral to each cavity than is found in the other abdominal segments.

The muscles of the segment, two of which in the other segments have their origins on the tergum, here originate entirely on the sternite. This anomaly may be due in part to a gradual thickening and deepening of the sternal sclerite resulting in a continual contact and subsequent fusion between the muscles involved and the upper part of the sternite.

Some of the interpretations of muscle action given by Schmidt appear to be incomplete when considered in relation to other muscles controlling the same joints and the probable functions of these muscles have been pointed out in appropriate places above.

The insertion of the remotor muscle on the first pleopod has been referred to as being on the "protopodite." The author, however, considers that as in the other pleopods, this muscle is inserted on a sclerite, which though greatly modified, is homologous with coxopodite "a" of the second pleopod. A more detailed study of this will be brought forth in a subsequent paper.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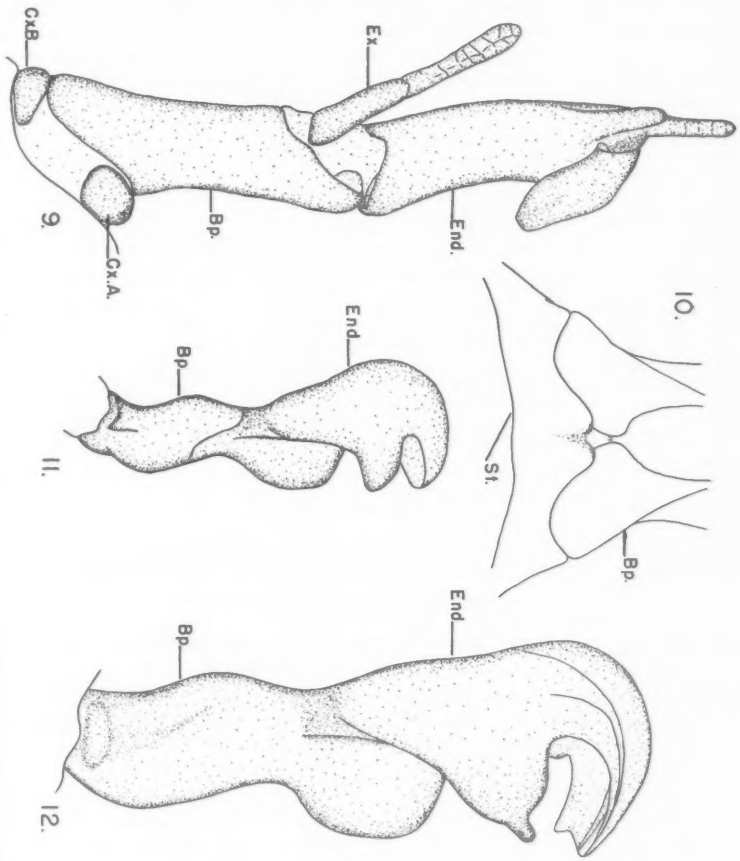
TER.
REM.
PL.
CX.
M.RED.B.
BP.
M.FLAG.EX.
EX.
M.FLAG.END.
END.
PL.
8
M.ROT.D.
ST.
M.ROT.V.
BP.



Explanation of Figures.—1. Ventral view of first and second abdominal segments. 2. Dorsal view of the first three abdominal segments and the last thoracic segment. Dorsal portion of the carapace and the terga are removed. Viscera and extensor muscles also removed. 3 and 4. Mesiocephalic views of the third abdominal segment of male. 5 and 6. Mesiocephalic views of the second abdominal segment of the male. Distal portion of the endopodite and exopodite not shown. 7 and 8. Cephalic views of sternum, pleuron, and proximal portion of pleo-

pods of the first abdominal segment of male. 9. Cephalic view of the second pleopod of male. 10. Caudal view of the sternum and the basal portions of the pleopods of the first abdominal segment of adult first form male. 11. Mesial view of left first pleopod of male (second form). 12. Mesial view of left first pleopod of male (first form).

a.c.	articular cavity
bp.	basipodite
cp.	carapace
cx.	coxopodite
cx.a.	coxopodite "a"
cx.b.	coxopodite "b"
end.-	endopodite
enp.s.a.	endophragmal skeletal apodeme
ex.	exopodite
M. abd. end.	*Musculus abductor endopoditis*
M. abd. ex.	*Musculus adductor endopoditis*
M. add. end.	*Musculus abductor endopoditis*
M. flag. end.	*Musculus flagellaris endopoditis*
M. flag. ex.	*Musculus flagellaris exopoditis*
M. red. b.	*Musculus reductor basipoditis*
M. rot. d.	*Musculus rotator dorsalis*
M. rot. v.	*Musculus rotator ventralis*
pl.	pleuron
pleo.	pleopod
rem.	*Musculus remotor*
st.	sternum
ter.	tergum
v.s.a.l. & 3.	*Musculus ventrales superficiales abdominis*
v.s.th.a.B.	*Musculus ventralis superficialis thoraco-abdominalis*

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ON THE PROPERTIES OF THE MULTIPLE COMPARISONS TEST[1,2]

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The multiple comparisons test is a procedure which has been recently proposed by the author (1951) for testing the differences among several treatments in an analysis of variance. The purpose of this paper is to give a more analytical discussion of the development and properties of this test.

STATEMENT OF THE PROBLEM

The problem may be stated in general terms as follows. Given a sample $x = (x_1, x_2, \cdots, x_n)$ where x_i is drawn at random from a normal population $n(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, n$, and given an estimate s^2 of the common variance σ^2, which is independent of x and is distributed as $\frac{1}{n_2}\sigma^2\chi^2_{n_2}$, where n_2, the "degrees of freedom" for s^2 is also known; a test is required for making a decision to accept one of the hypotheses in a system S defined in the following way. Let $\mu_{a_1}, \mu_{a_2}, \cdots, \mu_{a_n}$ be any one of the $n!$ permutations of the population means. Let

$$\underline{\mu_{a_1}\ \mu_{a_2}}, \underline{\mu_{a_3}}\ \mu_{a_4}, \underline{\mu_{a_5}}\ \underline{\mu_{a_6}\ \mu_{a_7}}\ \underline{\mu_{a_8}}\ \mu_{a_9}\ \underline{\mu_{a_{10}}\ \mu_{a_{11}}}, \cdots, \underline{\mu_{a_n}} \tag{1}$$

represent any plan of linking adjacent means together in groups of one, two, $\cdots$, n means so that no group is completely enclosed in another, varying from a plan with n single-mean groups to one consisting of only one group of all n means. Let H be the hypothesis stating that the left hand member of each pair of means not bracketed together in such a plan is less than the right hand member and that the relative rankings of each pair of means bracketed together is otherwise unrestricted. The system S consists of all distinct hypotheses which can be obtained in this way.

In the simple case $n = 2$, S contains the three hypotheses, μ_1, μ_2; μ_2, μ_1; and $\underline{\mu_1, \mu_2}$; that is, $\mu_1 < \mu_2$, $\mu_2 < \mu_1$, and μ_1 is unranked rela-

[1] This and the previous paper (1951) include and extend a thesis (1947) approved for the degree of Ph.D at the Iowa State College.

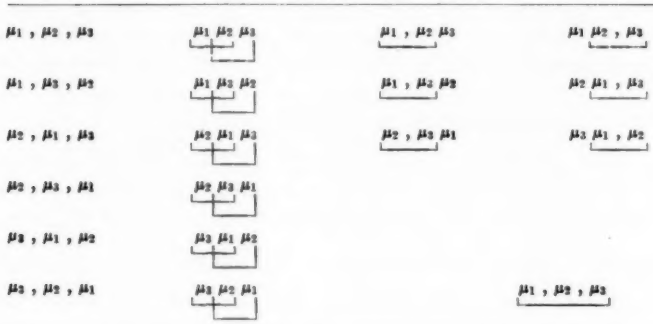
[2] For this and a companion paper [*Va. Jour. Sci.*, 2 (4)], presented at the Annual Meeting of the Academy, May 11, 1951, Mr. Duncan received the **J. Shelton Horsley Award.**

tive to μ_2. In the case $n = 3$, S contains the 19 hypotheses in table 1. The number of hypotheses in the system increases rapidly with n.

In the simple case $n = 2$, an ideal answer to the problem is provided by the symmetrical or two-sided t test. The test we seek for the case $n \geq 3$ may be regarded as a generalization in a sense of this well-known test.

TABLE 1

System of Hypotheses. Case $n = 3$



GEOMETRICAL NOTATION

In considering this problem it will be useful to represent the various comparisons within a sample $x = (x_1, x_2, \cdots, x_n)$ by a sample point $z = (z_1, z_2, \cdots, z_{n-1})$ in a sample space W_{n-1} with axes defined by $z_i = y_i/s$, where y_i is any set of standardized orthogonal comparisons, say $y_i = (\sum_{j=1}^{i} x_j - ix_{i+1})/\sqrt{i(i+1)}$, $i = 1, 2, \cdots, n-1$. Correspondingly any set of values for the comparisons among the population means $\mu_1, \mu_2, \cdots, \mu_n$ may be represented by a parameter point $\theta = (\theta_1, \theta_2, \cdots, \theta_{n-1})$ in a parameter space Ω_{n-1} with axes defined by $\theta_i = \eta_i/\sigma$, where

$$\eta_i = \left(\sum_{j=1}^{i} \mu_j - i\mu_{i+1}\right) \Big/ \sqrt{i(i+1)}, \qquad i = 1, 2, \cdots, n-1.$$

From these choices of axes it follows that the probability element at any sample point z in W_{n-1} is closely related to that of an F distribution with $n-1$ and n_2 degrees of freedom and is given by

$$f(z; \theta)\, dz = \frac{\Gamma\frac{1}{2}(n_1 + n_2)}{\Gamma(\frac{1}{2}n_2)(n_2\pi)^{\frac{1}{2}n_1}} \left(1 + \frac{n_1}{n_2} F\right)^{-\frac{1}{2}(n_1+n_2)} dz \qquad (2)$$

where F is the variance ratio, $\sum_{i=1}^{n_1} (y_i - \theta_i)^2/n_1 s^2$, and $n_1 = n - 1$. In the important special case $n_2 \to \infty$, (2) reduces to an $n - 1$ variate normal distribution with means θ, unit variances and zero correlations.

The subset of Ω_{n-1} defined by an element of the system S, say by $\mu_{a_1} \mu_{a_2}, \mu_{a_3} \cdots, \mu_{a_n}$ will be denoted by $\omega(a_1\, a_2, a_3 \cdots, a_n)$. The construction of any test T will consist of defining a region R $(a_1\, a_2, a_3 \cdots, a_n; T)$ corresponding to the subset $\omega(a_1\, a_2, a_3 \cdots, a_n)$, for all subsets in the system. The test T may then be described as the rule: if

$$z \in R\ (a_1\, a_2, a_3 \cdots, a_n; T)$$

make the decision D $(a_1\, a_2, a_3 \cdots, a_n)$, which consists of accepting the hypothesis:

$$H(a_1\, a_2, a_3 \cdots, a_n): \theta \in \omega(a_1\, a_2, a_3 \cdots, a_n).$$

In addition to the subsets in S we will also need to consider a class of "null" subsets:

$$\omega_0(a_1, a_2, \cdots, a_p): \mu_{a_1} = \mu_{a_2} = \cdots = \mu_{a_p},\ 2 \leq p \leq n,$$

and "critical" subsets

$$\omega_1(a_1, a_2, \cdots, a_p): \Omega_{n-1} - \omega_0(a_1, a_2, \cdots, a_p),$$

and their corresponding hypotheses, regions and decisions which will be denoted by $H_i(a_1, a_2, \cdots, a_p)$, $R_i(a_1, a_2, \cdots, a_p; T)$, and

$$D_i(a_1, a_2, \cdots, a_p),\ i = 0, 1.$$

Example 1: Figure 1 shows the parameter subsets and sample space regions for the familiar α-level two-sided t test in the case $n = 2$.

It will be noted in passing that a distinction is being made between the hypothesis $H(1, 2)$ which is accepted when the decision $D(1, 2)$ is made and the null hypothesis $H_0(1, 2): \mu_1 = \mu_2$ which is the central hypothesis

tested by this procedure. It will be agreed that acceptance of $H(1, 2)$ which merely states ignorance of the relative rankings of μ_1 and μ_2, is the more accurate interpretation of the decision made by an agronomist say, when he concludes that the observed means of two varieties are "not significantly different". This is not to say however that the null hypothesis $H_0(1, 2)$ is not the correct one to test. We are merely taking the usual view that insufficient evidence to reject this hypothesis is not sufficient evidence to accept it. The distinction is of little practical consequence in this case $n = 2$, but has to be made in all other cases $n > 2$ to avoid contradictions.

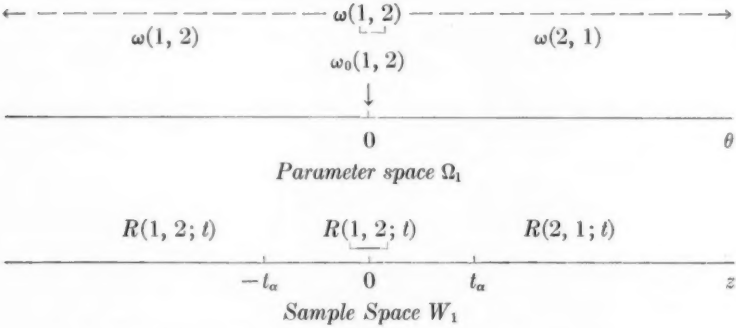


FIGURE 1. Parameter Subsets and Sample Space Regions, Case $n = 2$

The test boundaries $\pm t_\alpha$ in figure 1 are the usual α-level significant values of t, that is, they satisfy the relation $P[-t_\alpha < t_{n_2} < t_\alpha \mid H_0(1,2)] = \alpha$, where t_{n_2} is distributed as t with n_2 degrees of freedom.

Example 2. In the previous paper, Duncan (1951), the invalid procedure of applying a two-sided α-level t test jointly to the null hypothesis $H_0(i, j)$ for all ${}_nC_2$ combinations of i and j was termed an α-level multiple t test. Figures 2 and 3 show the parameter subsets and the sample space regions for the case $n = 3$ of this test.

Only six of the parameter subsets are shown explicitly in figure 2 and these are bounded by the lines $\mu_i = \mu_j$, $(i, j) = (1, 2), (1, 3), (2, 3)$. The remaining 13 subsets are obtained as combinations of the ones

illustrated. For examples, $\omega(1, 2\ 3) = \omega(1, 2, 3) + \omega(2,1, 3)$;

$\omega(1, 2, 3) = \omega(1, 2, 3) + \omega(2, 1, 3) + \omega(1, 3, 2)$; and $\omega(1, 2, 3) = \Omega_2$,

that is, the sum of all the subsets shown.

The boundaries of the 19 sample space regions in figure 3 are given by the six lines $(x_i - x_j)/\sqrt{2}s = \pm t$, $(i, j) = (1, 2), (1, 3), (2, 3)$.

WEIGHTING OF ERRORS AND DETERMINATION OF AVERAGE RISK

TWO-SIDED t TEST OF $H_0(1, 2)$.—In this section we will restrict ourselves mainly to the case $n = 3$, and will first consider the determination of an average weighted risk[3] for an α-level two-sided t test of the null hypothesis $H_0(1, 2)$. The parameter subsets in Ω_2 and sample space regions in W_2 for this test are the same as those shown in figure 1, except that the boundaries at the points $\theta = 0$ and $z = \pm t_\alpha$ in that figure will now be the lines $\theta_1 = 0$ and $z_1 = \pm t_\alpha$ respectively in the two dimensional spaces.

For the weighting of errors we will use the weight function $w_2(\theta, D)$ as defined in table 2, where $a(\theta_1)$ and $b(\theta_1)$ are monotone decreasing and monotone increasing functions of θ_1 over the ranges $\theta_1 \leq 0$ and $\theta_1 > 0$ respectively. The importance of the less conservative type-1-like error of deciding that a difference is positive when it is not, is measured by $a(\theta_1)$ and the importance of the more conservative type-2-like error of failing to decide that a difference is positive when it is, is measured by $b(\theta_1)$. Since the errors involved here do not coincide exactly with the Neyman and Pearson type 1 and type 2 errors we will refer to them as type a and type b errors.

Any one decision in the two-sided test may be said to commit none, either one or both types of errors. For the parameter values $\theta_1 < 0$ for example, if the decision $D(1, 2)$ is made no error is committed, hence the weight zero. If the decision $D(1, 2)$ is made a type b mistake is committed relative to $-\theta_1$, hence the weight $b(-\theta_1)$. If $D(2, 1)$ is made it may be considered that two errors are committed. One is the type a error of deciding that θ_1 is positive when it is not, the other is the type b error of failing to decide that $-\theta_1$ is positive as it is; hence the weight $a(\theta_1) + b(-\theta_1)$.

[3] See Wald (1939, 1942, and other publications).

Because of symmetry considerations it is appropriate to use the same weight functions with respect to $-\theta_1$ as with respect to θ_1.

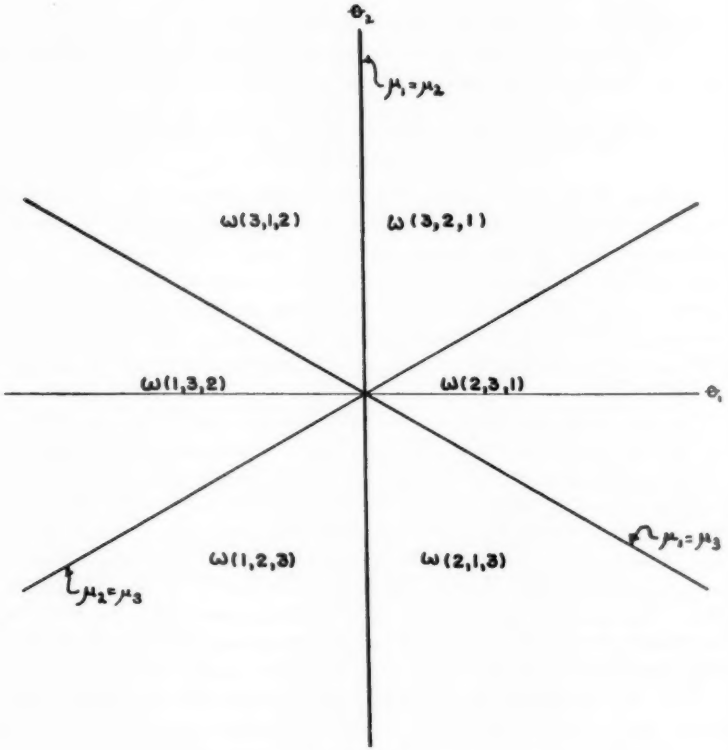


FIGURE 2. Subsets in Ω_2. Case $n = 3$

Using the above weighting function, the risk $r_2(\theta)$ for this two-sided test may be written as the sum of a type a and a type b risk as $r_a(\theta)$

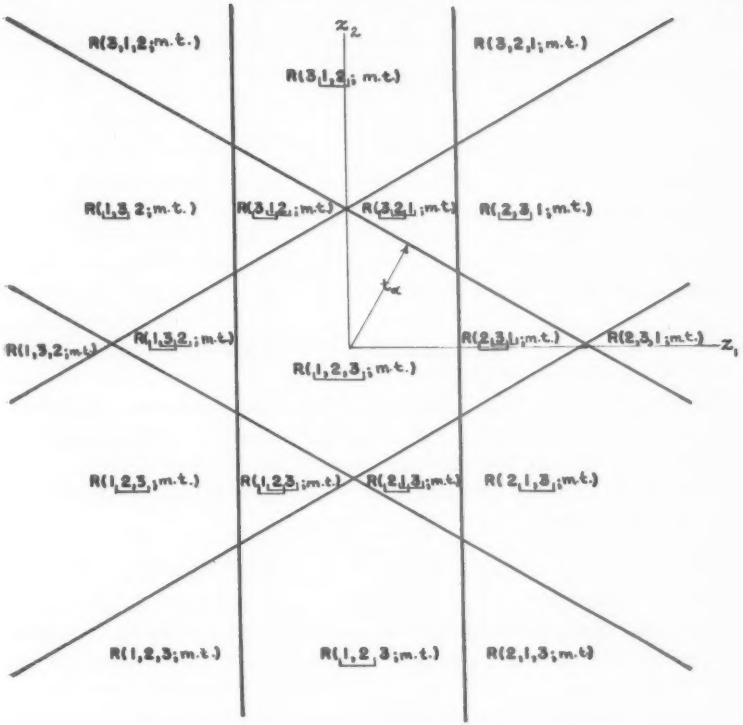


FIGURE 3. Regions for a Multiple t Test. Case $n = 3$

TABLE 2

Weight function $w_2(\theta, D)$ for two-sided test

Parameter Values	Decisions		
	$D(1, 2)$	$D(\underline{1, 2})$	$D(2, 1)$
$\theta_1 < 0$, i.e. $\mu_1 < \mu_2$	0	$b(-\theta_1)$	$a(\theta_1) + b(-\theta_1)$
$\theta_1 = 0$, i.e. $\mu_1 = \mu_2$	$a(0)$	0	$a(0)$
$\theta_1 > 0$, i.e. $\mu_2 < \mu_1$	$a(-\theta_1) + b(\theta_1)$	$b(\theta_1)$	0

$+ r_b(\theta)$, where

$$r_a(\theta) = a(\theta_1) \int_{R:\, z_1 > t_\alpha} f(z; \theta)\, dz,$$

$$r_b(\theta) = b(-\theta_1) \int_{R:\, z_1 > -t_\alpha} f(z; \theta)\, dz, \qquad \theta_1 < 0;$$

$$r_a(\theta) = 2a(0) \int_{R:\, z_1 > t_\alpha} f(z; \theta)\, dz, \qquad r_b(\theta) = 0, \qquad \theta_1 = 0; \quad (3)$$

$$r_a(\theta) = a(-\theta_1) \int_{R:\, z_1 < -t_\alpha} f(z; \theta)\, dz,$$

$$r_b(\theta) = b(\theta_1) \int_{R:\, z_1 < t_\alpha} f(z; \theta)\, dz, \qquad \theta_1 > 0.$$

If we now let $p_2(\theta)$ represent an appropriate parameter anticipated frequency function which will have the symmetry property $p_2(-\theta) = p_2(\theta)$, the average weighted risk A_2 can also be defined as the sum of two parts A_{2a} and A_{2b}, which may be termed a type a and a type b average risk. For these we may write

$$A_{2a} = \int_{\Omega_2} r_a(\theta) p_2(\theta)\, d\theta; \qquad A_{2b} = \int_{\Omega_2} r_b(\theta) p_2(\theta)\, d\theta. \qquad (4)$$

In terms of the above analysis the motivation for choosing an α-level two-sided t test for a given requirement may be based on the following two points:

(1) For all α-level tests, and for all symmetrical weight and parameter anticipated frequency functions of the above kind, the two-sided t test gives the minimum type a average risk A_{2a}, the minimum type b average risk A_{2b} and hence the minimum combined average risk A_2.

(2) For all two-sided t tests, and for the particular forms of $w_2(\theta, D)$ and $p_2(\theta)$ involved, the test at the chosen level α minimizes A_2 with respect to changes in α.

With regard to the first point it will be noted that the minimization of A_{2b} is equivalent to the familiar property of uniform maximum power relative to all unbiased tests. Observance of the second point is restricted in practice by limitations on the range of significance levels for tabled values of t and also by insufficient consideration of $a(\theta)$ and $b(\theta)$ and inadequate knowledge of $p_2(\theta)$. Nonetheless it will be agreed that in choosing a significance level the aim should be to pick a value such that

if a lower, more conservative one were taken, the increase in A_{2b} would exceed the decrease in A_{2a} ; while on the other hand, if a higher, less conservative level were taken, the increase in A_{2a} would exceed the decrease in A_{2b} .

MULTIPLE t TEST.—We will now consider the determination of an average weighted risk for the multiple t test. All of the points arising in this are applicable also in the multiple comparisons test with which we are ultimately concerned. The weight function to be proposed can be defined completely, except for one important modification to be treated later, as the reduced sum of three two-sided test average risks in the following way:

First, each of the 19 decisions may be treated as the sum of three sub-decisions with respect to each of the three differences between the means considered a pair at a time. On weighting the errors accordingly we may write

$$w_3(\theta, D(z)) = \frac{2}{3} \sum_{ij} w_2(\lambda_{ij}, D(d_{ij})), \qquad ij = 12, 13, 23. \tag{5}$$

where $\lambda_{ij} = (\mu_j - \mu_i)/\sqrt{2}\sigma$ and $d_{ij} = (x_j - x_i)/\sqrt{2}s$ and $D(d_{ij})$ consists of the three decisions:—$D(i, j)$: decide $\mu_i < \mu_j$, $D(\underline{i, j})$: decide μ_i cannot be ranked relative to μ_j , $D(j, i)$: decide $\mu_j < u_i$. The reduction factor $2/3$ is introduced in allowance of the fact that these differences are correlated in such a way that three may be considered as equivalent to only two independent differences. Thence, we will obtain a risk function $r_3(\theta)$ which is the reduced sum of three two-sided test risk functions,

$$\begin{aligned} r_3(\theta) &= r_{3a}(\theta) + r_{3b}(\theta) \\ &= \frac{2}{3} \sum_{ij} r_{2a}(\lambda_{ij}) + \frac{2}{3} \sum_{ij} r_{2b}(\lambda_{ij}), \qquad ij = 12, 13, 23. \end{aligned} \tag{6}$$

Next, in considering a parameter anticipated frequency function $p_3(\theta)$ it will be appropriate from symmetry considerations to choose one which gives the same symmetrical marginal distributions $p_2(\lambda_{ij})$ for each of the differences λ_{ij} . That is, to have $p_2(\lambda_{12}) = p_2(\lambda_{13}) = p_2(\lambda_{23}) = p_2(-\lambda_{12}) = p_2(-\lambda_{13}) = p_2(-\lambda_{23})$. Then on integrating the risk over the parameter space we obtain an average risk A_3 which is the sum of the average risks of two two-sided tests,

$$A_3 = A_{3a} + A_{3b} = 2A_{2a} + 2A_{2b} . \tag{7}$$

But for the required modification, we could say that the multiple t

test had identical properties with those of two independent two-sided t tests and thus had a minimum average risk.

The necessity for the modification is most evident when we consider the properties of this procedure as a test of the three-mean null hypothesis $H_0(1, 2, 3)$, that is, $H_0: \mu_1 = \mu_2 = \mu_3$. It is only in the region $R(\underline{1, 2, 3}; m.t.)$ that this hypothesis is not rejected. It can be readily shown that

$$P[z \in R(\underline{1, 2, 3}; m.t.) \mid H_0(1, 2, 3)] = \int_{-t_\alpha}^{t_\alpha} f(q_{3,n_2})\, dq \qquad (8)$$

where $f(q_{3,n_2})$ is the probability density of the studentized range of a sample of size 3 with n_2 degrees of freedom. This probability is far too small and the complementary probability, α_3, of wrongly rejecting $H_0(1, 2, 3)$ is far too large to be consistent with the choice of α as the allowable probability, α_2, for wrongly rejecting any one of the two-mean null hypotheses $H_0(1, 2)$, $H_0(1, 3)$ and $H_0(2, 3)$. For example, in the case $n_2 \to \infty$ and $\alpha = .05$, we find that $\alpha_3 = .1223$ which is an unduly large figure relative to .05.

It can be argued[4] that since $H_0(1, 2, 3)$ is the null hypothesis that two independent differences are each zero that α_3 should be made equal to $1 - (1 - \alpha)^2$ to be consistent with the choice of α for α_2.

A relevant graphical picture can be obtained by constructing the critical boundaries in W_2 for two α-level two-sided t tests of the independent hypotheses $\mu_1 = \mu_2$ and $\mu_1 + \mu_2 = 2\mu_3$ in the case $n_2 \to \infty$ and $\alpha = .05$. These are shown in figure 4. The argument is that the null region $R(\underline{1, 2, 3}; m.t.)$ should be as large as the null region $R(\underline{1, 2, 3}; t^2)$ in this figure. In very loose terms we can say that the attempt by the multiple t test to test three differences, where only two independent ones are present, is causing overcrowding at the intersections of the respective null regions $R(\underline{1, 2}; t)$, $R(\underline{1, 3}; t)$ and $R(\underline{2, 3}; t)$.

Weighting for Overcrowding: The problem of including the above considerations in the overall average risk function is quite an elusive and complex one and the following approach may or may not need one or two minor adjustments in the light of further thought on the subject. It does however appear to be essentially correct with respect to the main issues involved and to lead to conclusions which are not likely to be changed by any of the possible modifications.

[4] See companion paper Duncan (1951) section entitled 'Consistent significance levels'.

First, on the circumference of any circle $r^2 = z_1^2 + z_2^2$ in W_2 we will define an overcrowding factor $k(r)$ as the ratio $g(r)/h(r)$ where $g(r)$ is the length of arc contained in the intersection of $R(1, 2, 3; t^2)$ and

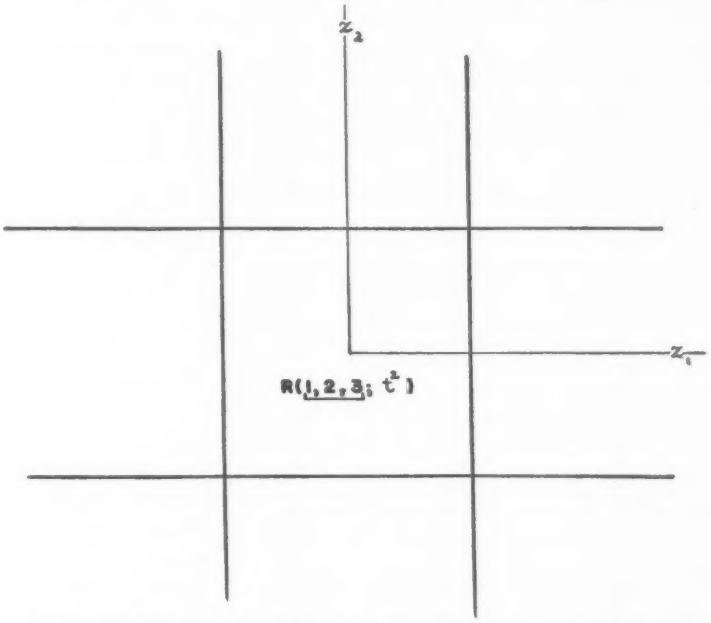


FIGURE 4. Sample Space Regions for Two Independent α-Level Two-Sided t Tests.

$R_1(1, 2, 3; m.t.)$, $h(r)$ is the length of arc contained in R_1 $(1, 2, 3; m.t.)$ and R_1 $(1, 2, 3; m.t.)$, as indicated before, is the critical region $W_2 - R(1, 2, 3; m.t.)$. Thus we obtain

$$k(r) = 8r\left(\cos^{-1}\frac{t_\alpha}{r}\right)\Big/ 12r\left(\cos^{-1}\frac{t_\alpha}{r}\right) = \frac{2}{3}, \qquad t_\alpha < r < \frac{2}{\sqrt{3}}\,t_\alpha,$$
$$k(r) = 8r\left(\sin^{-1}\frac{t_\alpha}{r} - \frac{\pi}{4}\right)\Big/ 2\pi r, \qquad \frac{2}{\sqrt{3}}\,t_\alpha < r < \sqrt{2}t_\alpha, \tag{9}$$

and $k(r)$ is zero for all other values of r.

Next, for any given sample point z, let ZO be the line joining z to the origin O, and AB the line passing through O at right angles to ZO. Let $A'B'$ be a line in Ω_2 corresponding to AB in W_2. That is, if $z_2 = Kz_1$ is the equation for AB then $\theta_2 = K\theta_1$ is the equation for $A'B'$. Let $\theta B'$ be the perpendicular from any given parameter point θ to the line $A'B'$ and put $\pm\lambda$ for the length of $\theta B'$, using $+$ or $-$ depending on whether θ is on the same side of $A'B'$ as z is of AB or the reverse side.

If the decision at z is $D_1(1, 2, 3)$ or $D_0(1, 2, 3)$, (that is, if it rejects or fails to reject $H_0(1, 2, 3)$), it may be said to imply that the sign of λ is positive or unknown respectively. With this in mind we may adopt a weight function for overcrowding as defined in table 3.

TABLE 3

Weight function for overcrowding.

Values of λ	Decision	
	$D_0(1, 2, 3)$	$D_1(1, 2, 3)$
$\lambda \leq 0$	0	$k(r) \cdot a(\lambda)$
$\lambda > 0$	$-k(r)b(\lambda)$	0

In doing this we are taking the attitude for example that if λ is not positive and $D_1(1, 2, 3)$ is made, an additional type a error is committed, the importance of which depends on the degree of overcrowding at z as measured by $k(r)$ and on the magnitude of λ as measured by $a(\lambda)$. On the other hand, if λ is positive and $D_1(1, 2, 3)$ is made, this will be a correct decision and will offset some of the type b errors in the rest of the test. The importance of this is also being taken to depend on the overcrowding and on the magnitude of λ as measured by $b(\lambda)$.

Modification in Average Risk due to Overcrowding.—The calculation of the modification in average risk due to the overcrowding discussed above can be simplified considerably by assuming a spherically symmetrical parameter anticipated frequency function $p_3(\theta)$. Since $p_3(\theta)$ is already restricted by the nature of the problem to be symmetrical with respect to each of the three differences involved, and since these modifications are only small relative to all of the risks concerned, it will be seen that this is not likely to lead to appreciable errors.

Under this assumption it follows that the required modification in average risk, which we will denote by A_3', can be expressed in terms of the probability densities on the line $z_1 = 0$ and within the interval

$t_\alpha < z_2 < \sqrt{2}t_\alpha$. It may be written as the sum of a type a component A'_{3a} and a type b component A'_{3b} defined by

$$A'_{3a} = \int_{\theta_2 \leq 0} a(\theta_2) \int_R g(r) f(z; \theta)\, dz p_3(\theta)\, d\theta$$
$$A'_{3b} = -\int_{\theta_2 > 0} b(\theta_2) \int_R g(r) f(z; \theta)\, dz p_3(\theta)\, d\theta \tag{10}$$

and R is the given region: $z_1 = 0$, $t_\alpha < z_2 < \sqrt{2}t_\alpha$. Thence the final average risk for the multiple t test is

$$A_3 + A'_3 = (2A_{2a} + A'_{3a}) + (2A_{2b} - A'_{3b}). \tag{11}$$

The main problem at hand is to modify this test so as to reduce the type a risk at least back to $2A_{2a}$ with a smaller increase taking place in type b risk bringing it back towards $2A_{2b}$.

THE MULTIPLE COMPARISONS TEST.—In the proposed multiple comparisons test, the hexagonal boundaries of the null region $R(\underline{1, 2, 3}; m.t.)$ are expanded outwards to the likelihood ratio circular boundary $r^2 = 2F_{2,n_2,\alpha_3}$, where F_{2,n_2,α_3} is the α_3-level significant value of F for 2 and n_2 degrees of freedom, and $\alpha_3 = 1 - (1 - \alpha)^2$. The regions of the new test obtained in this way are illustrated in figure 5.

The six small regions lying outside the new null region $R(\underline{1, 2, 3}; m.c.)$ but inside the previous one $R(\underline{1, 2, 3}; m.t.)$ require special mention. In each of these we do not make any one of the eighteen non-null decisions belonging to the system S. However, for any sample point z in one of these regions we make the new decision that λ is positive, where λ is defined[5] as above, relative to z. A decision of this type may be conveniently left in the form that two of the means considered as a group are higher or smaller than the third. If z falls in the region at the top of $R(\underline{1, 2, 3}; m.c.)$ for example, the decision would be that μ_1 and μ_2 as a group are higher than μ_3. A convenient notation for this region is $R(\overline{3, 1, 2})$. The other five regions of this kind are denoted in a similar way. The remaining eighteen non-null regions of this test are identical with those of the multiple t test except for the new circular boundaries that six of them have with the null region $R(\underline{1, 2, 3}; m.c.)$.

[5] It may be noted that $\lambda = E(c)$ where c is as defined in the preceding paper (1951), page 185.

Using a similar approach to determine the average risk for this test, it is necessary to introduce a compensation-for-overcrowding factor $k^*(r)$ defined as the ratio $g^*(r)/h^*(r)$, where $g^*(r)$ is the length of arc on the circumference of the circle $r^2 = z_1^2 + z_2^2$ contained in the intersec-

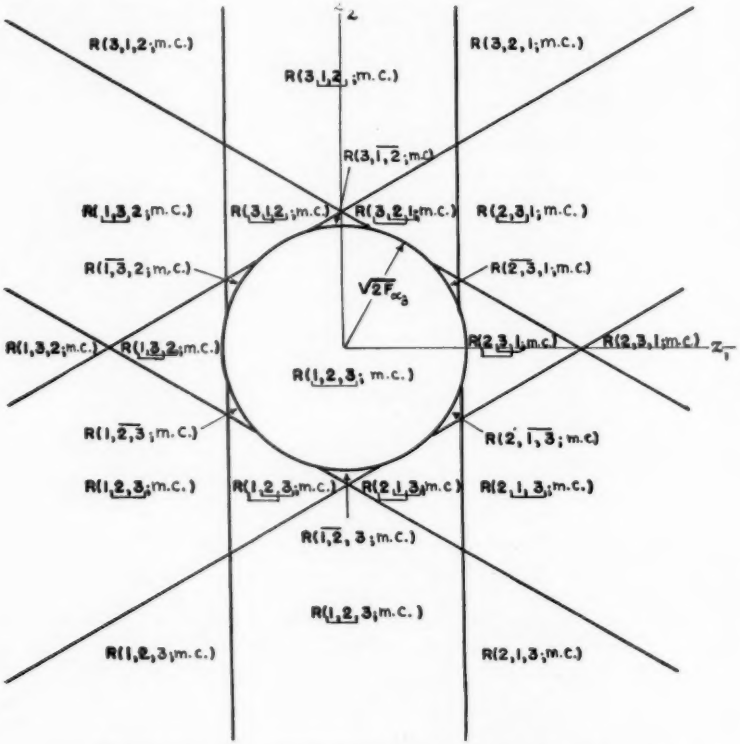


FIGURE 5. Regions for Multiple Comparisons Test. Case $n = 3$

N.B. The six regions R(3,1,2;m.c.), etc. have been exaggerated for clarity by decreasing the size of R(1,2,3;m.c.)

tion of $R(1, 2, 3; m.c.)$ and $R_1(1, 2, 3; t^2)$ and $h^*(r)$ is the length of arc contained in $R_1(1, 2, 3; t^2)$.

For overcrowding, the factor is

$$k(r) = 8r\left(\sin^{-1}\frac{t_\alpha}{r} - \frac{\pi}{4}\right)\Big/ 2\pi r, \qquad \sqrt{2F_{2,n_2,\alpha_3}} < r < \sqrt{2}t_\alpha \quad (12)$$

and zero for other values of r. For compensation for overcrowding, the factor is

$$k^*(r) = \frac{8r\left(\sin^{-1}\frac{t_\alpha}{r} - \frac{\pi}{4}\right)}{8r\left(\sin^{-1}\frac{t_\alpha}{r} - \frac{\pi}{4}\right)} = 1, \qquad t_\alpha < r < \sqrt{2F_{2,n_2,\alpha_3}} \quad (13)$$

and zero for all other values of r.

On incorporating both the overcrowding factor $k(r)$ and the compensation factor $k^*(r)$ into the average weighted risk in the same way as $k(r)$ was used in the previous test, we conclude that the complete value for this test is

$$\begin{aligned} A_3 = 2A_2 + A_3' - A_3^* = A_{3a} + A_{3b} = {} & (2A_{2a} + A_{3a}' - A_{3a}^*) \\ & + (2A_{2b} + A_{3b}' - A_{3b}^*) \end{aligned} \quad (14)$$

where

$$\begin{aligned} A_{3a}' &= 4\int_{\theta_2 \le 0} a(\theta_2)P'(\theta)p_3(\theta)\,d\theta, \\ A_{3b}' &= -4\int_{\theta_2 > 0} b(\theta_2)P'(\theta)\,d\theta, \\ P'(\theta) &= \frac{1}{4}\int_{R'} g(r)f(z;\theta)\,dz, \end{aligned} \quad (15)$$

R' is the region: $z_1 = 0$, $\sqrt{2F_{2,n_2,\alpha_3}} < z_2 < \sqrt{2}t_\alpha$,

$$\begin{aligned} A_{3a}^* &= 4\int_{\theta_2 \le 0} a(\theta_2)P^*(\theta)p_3(\theta)\,d\theta, \\ A_{3b}^* &= -4\int_{\theta_2 > 0} b(\theta_2)P^*(\theta)p_3(\theta)\,d\theta, \\ P^*(\theta) &= \frac{1}{4}\int_{R^*} g^*(r)f(z;\theta)\,dz \end{aligned} \quad (16)$$

and R^* is the region: $z_1 = 0$, $t_\alpha < z_2 < \sqrt{2F_{2,n_2,\alpha_3}}$.

The complete evaluation of the average weighted risk depends, of course, on further knowledge of the weight functions $a(\theta_2)$ and $b(\theta_2)$ and on the parameter function $p_3(\theta)$. However, much can be done without

this by expressing $2A_{2a}$ and $2A_{2b}$ in forms similar to (15) and (16) and considering the values of the known parts of the complete integrand separately for each value of θ. Thus we may write

$$2A_{2a} = 4 \int_{\theta_2 \leq 0} a(\theta_2) P(\theta) p_3(\theta)\, d\theta$$

and

$$2A_{2b} = 4 \int_{\theta_2 > 0} b(\theta)\, [1 - P(\theta)] p_3(\theta)\, d\theta, \tag{17}$$

where

$$P(\theta) = \int_R f(z; \theta)\, dz$$

and R is the region: $z_2 > t_\alpha$. Thence we have

$$A_{3a} = 4 \int_{\theta_2 \leq 0} a(\theta_2)[P(\theta) + P'(\theta) - P^*(\theta)]\, p_3(\theta)\, d\theta$$

and

$$A_{3b} = 4 \int_{\theta_2 > 0} b(\theta_2)[1 - P(\theta) - P'(\theta) + P^*(\theta)]\, p_3(\theta)\, d\theta \tag{18}$$

The fairly close agreement between the average weighted risk for this test and that of two independent two-sided t tests can be seen from a consideration of the smallness of the differences $P'(\theta) - P^*(\theta)$ relative to $P(\theta)$ for $\theta_2 \leq 0$ or to $1 - P(\theta)$ for $\theta_2 > 0$. These are shown for example in table 4 for a few selected values of θ in the special case $n = 3$, $\alpha = .05$, $n_2 \to \infty$.

From these figures and a consideration of similar details in other cases, it will be seen that the type a average risk A_{3a} is even smaller than $2A_{2a}$ and at the same time A_{3b} is also smaller than $2A_{2b}$.

On further consideration of the mechanics of the above weighting system it appears that by choosing the likelihood ratio boundary for the region $R(1, 2, 3; m.c.)$ we have succeeded in maximizing the differences $P'(\theta) - P^*(\theta)$ for all values of θ and thus in minimizing the average risk with respect to tests having the same size α_3 for the null region $R(1, 2, 3)$. Furthermore, since $A_2 = A_{2a} + A_{2b}$ is the minimum risk for a single two-sided t test and since $2A_{2a} - A_{3a}$ and $2A_{2b} - A_{3b}$ are both small and appear to be at least roughly equivalent reductions, $A_3 = A_{3a} + A_{3b}$ should be fairly close to the absolute minimum possible for the required test.

TABLE 4

P values.[6] Multiple Comparisons Test

Case $n = 3$, $\alpha = .05$, $n_2 \to \infty$

θ_2 \ θ_1		0	±1	±5
1.5	$1 - P(\theta)$	.6772	.6772	.6772
	$-P'(\theta)$	−.0467	−.0283	.0000
	$P^*(\theta)$	.0317	.0195	.0000
	$P^* - P'$	−.0150	−.0088	.0000
0	$P(\theta)$	.0250	.0250	.0250
	$P'(\theta)$	.0044	.0027	.0000
	$-P^*(\theta)$	−.0044	−.0027	.0000
−.5	$P(\theta)$	.0069	.0069	.0069
	$P'(\theta)$	.0012	.0007	.0000
	$-P^*(\theta)$	−.0014	−.0008	.0000
	$P' - P^*$	−.0002	−.0001	.0000

[6] Evaluated using tables (1942) and (1945).

TABLE 5

P values. Multiple Range Test

Case $n = 3$, $\alpha = .05$, $n_2 \to \infty$

θ_2 \ θ_1		0	±1	±5
1.5	$-P'(\theta)$	−.0478	−.0290	.0000
	$P^*(\theta)$	.0367	.0223	.0000
	$P^* - P'$	−.0111	−.0067	.0000
0	$P'(\theta)$	.0047	.0029	.0000
	$-P^*(\theta)$	.0047	.0029	.0000
−.5	$P'(\theta)$	.0013	.0008	.0000
	$-P^*(\theta)$	.0014	.0009	.0000
	$P' - P^*$	−.0001	.0001	.0000

For illustrative purposes, a set of figures comparable to those in table 4 have been determined for the multiple range test obtained by expanding $R(1, 2, 3; m.t.)$ to the hexagonal boundaries[7] $(x' - x'')/s = q_{3,\infty,.0975} = 2.91849$, where x' and x'' denote the highest and lowest of the three variates x_1, x_2, x_3, and $q_{3,\infty,.0975}$ is the .0975 level significant value of the range[8] of three variates in the case $n_2 \to \infty$. These are given in table 5. The superiority of the multiple comparisons test follows from comparing the differences $P'(\theta) - P^*(\theta)$ for the two tests.

MULTIPLE COMPARISONS TEST FOR ALL CASES

The multiple comparisons test for any value of n may be defined by a recursion method for constructing its sample regions in W_{n-1}.

To construct the regions for an n variate test: (a) Consider the n possible combinations of the variates taken $n - 1$ at a time; (b) Divide W_{n-1} into n intersecting systems of regions, each representing an $n - 1$ variate test for each of the n combinations. The regions formed by the intersections of these n systems are the required regions except for one modification which follows; (c) Adjust the null region $R(1, 2, \cdots, n)$ to have the boundary: $\Sigma z_i^2 = (n - 1)F_{n-1,n_2,\alpha_n}$ where the latter[9] is the α_n-level significant value of F with $n - 1$ and n_2 degrees of freedom respectively and α_n is the "α-per-degree-of-freedom" level $1 - (1-\alpha)^{n-1}$. The naming of, and thus the decision which goes with, each region formed in this way, will follow readily from the regions for which it is the intersection, except for certain new regions such as $R(3, 1, 2)$ in the case $n = 3$. These are the regions which fall inside the unadjusted null region $R(1, 2, \cdots, n)$ but outside its adjusted and final form $R(1, 2, \cdots, n; m.c.)$. When a sample point z falls in one of these regions, we decide to accept the hypothesis $H: \lambda > 0$ where λ is defined relative to z as in the section on weighting for overcrowding.

The multiple comparisons test is identical with the two-sided t test in the case $n = 2$. From this we may use the recursion method to con-

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[7] The new null region obtained in this way may be denoted by $R(1, 2, 3, ; m.r)$ and has the same size as $R(1, 2, 3, ; m.c.)$ in figure 5.

[8] See Pearson and Hartleys' Tables (1941).

[9] Evaluated by using inverse interpolation (see Whittaker and Robinson, 1934) in K. Pearson's tables (1934a and 1934b).

struct the three-variate test as given in section 4.3, then from this, the four-variate test, and so on.

The procedure of weighting for crowding and compensation for overcrowding will be seen to generalize fairly readily from the case $n = 3$ to the cases $n > 3$. From this it seems reasonable to conclude that the average risk of the multiple comparisons test for the general case is fairly close to being a minimum.

CONCLUDING REMARKS

No attempt has been made in this paper to illustrate the various issues involved with examples from applied problems, or, at the end, to present the proposed test in a form convenient for practical applications. This has been done already in the preceding paper by the author (1951) and the reader is asked to refer to this for a discussion stressing the more applied aspects.

The work in this and the preceding paper has been done with the financial assistance of a University of Sydney Pawlett Scholarship at the Iowa State College, a Commonwealth Research Grant at the University of Sydney and a contract with the Bureau of Agricultural Economics at the Agricultural Experiment Station of the Virginia Polytechnic Institute.

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Observations on the Egg Masses of Spotted Salamanders, *Ambystoma maculatum* (Shaw), in the Williamsburg Area

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Reported here are observations on the nesting of Spotted Salamanders, *Ambystoma maculatum* (Shaw), on the Virginia coastal plain. The data on the time of egg deposition, the number of eggs voided in discrete groups, and the dimensions of egg masses are compared with published records of the nesting of this species in other regions.

On February 13, 1949 nests of *Ambystoma maculatum* were discovered for the first time in Tidewater Virginia by Dr. J. T. Baldwin and Dr. B. M. Speese of the College of William and Mary. They collected several egg masses and adult specimens from roadside ditches bordering State Highway 5 about 2 miles west of Five Forks, James City County. Subsequent field work by the senior author disclosed a number of other breeding habitats in the Williamsburg area. One of these was noteworthy for its small water area and large number of egg masses. Excavation of a clay bank adjacent to the Confederate Redoubt one mile southeast of Williamsburg resulted in the formation of two pools. These pools are shallow (maximum depth, 12 inches), and have a combined surface area of less than 300 square feet, yet in 1951 they contained 47 discrete egg masses.

In any locality the date on which egg-deposition is initiated varies from year to year. Brimley (1939) reports that in the vicinity of Raleigh, North Carolina, breeding begins in late January in some years, and as late as March in others. This great latitude in the date on which egg-laying is initiated has not been confirmed by the observations of Wright (1908) and Bishop (1941), who found that in western New York, over a period of many years, the first egg masses of *Ambystoma maculatum* have invariably been noted between late March and mid-April. In the Williamsburg area eggs were first noted on February 13th in 1949, and on February 24th in 1951.

In habitats near Williamsburg the egg-laying period of *Ambystoma maculatum* coincided with that of *Hemidactylium scutatum* in 1951, lasting from February 24th to March 10th. In some habitats both of

1 Work on this report was completed when the senior author was a staff member of the Virginia Fisheries Laboratory and the Department of Biology of the College of William and Mary.

these species were encountered, but this was not usual. In a pond at Five Forks *Hemidactylium* females and eggs were found in the moss, *Thuidium delicatulum* (Hedw.) Mitt. along the banks, and masses of *Ambystoma maculatum* eggs were noted nearby attached to submerged sticks under from four to ten inches of water.

The eggs of *Ambystoma maculatum* are deposited in a matrix of jelly. The dimensions of an egg mass are partly a function of the number of eggs involved, but are also influenced by the amount of water which has been absorbed by the eggs and jelly, and by the extent of cohesion of the jelly matrix. Bishop (1941) noted that in New York egg masses in early stages of development had diameters of from 2½ to 3½ inches, but he pointed out that continued water absorption and partial disintegration of the matrix may greatly increase the diameter, ". . . one such distended mass measured 4½ inches in diameter." Many Virginia egg masses were larger than those reported by Bishop. The smaller masses tended to remain spherical when suspended in water, but the larger aggregations of eggs appeared as elongated oval masses. The largest Virginia egg mass measured 156 mm. long, 119 mm. wide, and 43 mm. deep; the smallest mass measured 76 mm., 63 mm., and 28 mm. for the respective dimensions.

Bishop (1941) found that the ovaries of a gravid *Ambystoma maculatum* may contain over 200 eggs, and that most nests contain less eggs than the probable full ovarian complement of a female. Although some nests contained 200 eggs, others contained as few as 12, with the number generally nearer 100 (Bishop, 1941; Pope, 1944). Bishop (1943) reports that a female may deposit her eggs in one or two large masses, or in a half dozen small ones. These reports indicate that an egg-group found in the field is not likely to be correlated with the number of ovarian eggs in the gravid female that deposited it. The possibility that more than one female may deposit their eggs in a single nest is suggested by Hay's (1893) report of egg masses containing from 300 to 400 eggs. It is also possible that this number of eggs was arrived at by estimate rather than through a direct count, or that aberrant specimens were involved.

Twenty-one nests from the Williamsburg area were dissected to determine the numbers of eggs involved. These nests contained the following numbers of eggs: 190, 164, 151, 151, 150, 129, 125, 125, 124, 122, 113, 96, 89, 88, 81, 81, 77, 73, 57, 52, and 49. The mean of these counts is 109 eggs, with a standard deviation of 38.2. In nine New York nests Bishop (1941) found the following numbers of eggs: 256, 167, 159, 144, 124, 120, 101, 75, and 61. The mean number of eggs in these nine nests is 134, and the standard deviation is 54.7. These data show that the number of eggs deposited by *Ambystoma maculatum* in a discrete mass is highly variable. Smith's (1950) report that in Kansas each nest contains "approximately 100 eggs" is at variance with these findings.

SUMMARY

In the Williamsburg area egg masses of *Ambystoma maculatum* were first noted in middle and late February, in 1949 and 1951. In the latter year, in the ponds near Confederate Redoubt located one mile southeast of Williamsburg, the last egg masses had been deposited by March 10th, the term of egg-laying lasting two weeks. Eggs were counted in 21 nests, and their numbers per mass ranged from 49 to 190. Little central tendency was evident from these data. Egg masses were variable in size and shape, with smaller ones usually appearing spheroidal, and larger ones frequently elongate and flattened.

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The Production of Hyperkeratosis by the Administration of a Lubricant

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Since the report on bovine hyperkeratosis (X-Disease) by Olafson (1947), observations of various workers have indicated that the condition is widespread in the United States. The disease has appeared on a number of farms in Virginia, and in some instances has caused considerable losses. The etiology of many outbreaks has remained obscure in spite of attempts to identify an infectious agent, a nutritional disturbance, or a toxic substance. During the investigation of an outbreak in March 1951 it became apparent that a toxic substance or substances might be the causative agent. Materials were obtained and experiments conducted to test this possibility. This report describes the results obtained with one of the materials, a lubricant, with which symptoms and lesions of bovine hyperkeratosis were experimentally produced in calves.

LITERATURE

The early attempts to demonstrate the etiological agent, recently reviewed by Olson and Cook (1951), either failed or resulted in the production of one of two conditions, both unlike the spontaneously occurring disease. The early attempts at experimental production of bovine hyperkeratosis failed to produce the characteristic lacrimation, skin lesions, or the typical chronic symptoms. Olson, Cook and Brouse (1950) reported on an outbreak in a group of calves fed various rations. They obtained data that indicated an unknown factor or factors in the particular alfalfa hay and dicalcium phosphate used that enhanced the severity of the disease. Olson and Cook (1951) later reported the development of hyperkeratosis in experimental calves fed the feed-stuff left over from the original feeding trial. The environment and contact were eliminated as possible factors. Olafson and McEntee (1951) were able to produce experimentally bovine hyperkeratosis by the feeding of a processed concentrate feed-stuff. Later McEntee, Hansel, and Olafson (1951) succeeded in producing hyperkeratosis in cattle with the free fatty acid fraction of the ether extract of the processed concentrate. Wagener (1951) published recently the results of his observations made from 1946 to 1948 on bovine hyperkeratosis in Germany. He was able to produce the disease with a particular wood preservative which

Received for publication November 30, 1951.

affected the animals by contact, feeding, or inhalation. These recent developments indicate that bovine hyperkeratosis is caused by a toxic substance or substances, although the specific agent has not yet been reported.

MATERIALS AND METHODS

Approximately five pounds of lubricant (Lot 1) were obtained from a farm on which hyperkeratosis had occurred. Normal calves, ranging in age from 33 to 65 days, were obtained from the Virginia Polytechnic Institute herds and placed in a barn in separate stalls. No cattle had been housed in this barn for the past three years. The ration consisted of hay, grain, and milk obtained from the parent herd in which hyperkeratosis is known not to have occurred. The calves were all handled in the same manner, except for exposure to the lubricant, and no attempt at isolation was made. Four calves were administered varying amounts of lubricant by means of gelatin capsules; two calves were given varying amounts of a lubricant (Lot 2) purchased on the market, and two calves served as controls. One control calf (No. 241) was kept in a separate stall while the other control (No. 1356) was kept for a total of 65 days in a stall with calves receiving the lubricant. The rate and duration of administration of the lubricant were varied according to the response of the calf. In all cases the administration was stopped when it became obvious that irreparable damage had occurred.

The lubricant was administered according to the following schedules:

Calf 242—Lot 1 lubricant. 15 grams daily for the first 7 days, none for the next 5 days, then 10 grams daily for the next 12 days of administration.

Calf 1351—Lot 1 lubricant. 15 grams daily for the first 13 days, 10 grams on the 14th day, then 15 grams on the 15th and 16th days.

Calf 247—Lot 1 lubricant. 15 grams daily for the first 13 days, 10 grams daily for the next 4 days, then 5 grams on the 18th day of administration.

Calf 252—Lot 1 lubricant. 5 grams daily for the first 10 days, then 5 grams on the 12th, 14th, 16th and 17th days.

Calf 1366—Lot 2 lubricant. 10 grams daily for the first 12 days, none on the 13th day, then 10 grams daily for the next 5 days.

Calf 1371—Lot 2 lubricant. 15 grams daily for the first 12 days, none on the 13th day, then 15 grams daily for the next 5 days.

RESULTS AND DISCUSSION

The four calves exposed by means of oral administration to the lubricant obtained from the farm and the two calves similarly exposed to the purchased lubricant developed symptoms and lesions associated

with bovine hyperkeratosis; no symptoms or lesions developed in the controls. During the period of study, July 10, 1951 to October 25, 1951 and subsequently, no symptoms or lesions of hyperkeratosis and no evidence of a dietary deficiency appeared in 35 calves in the parent herd which received the same ration as the calves used in the experiment.

Since the response of the calves exposed to the lubricant obtained from the farm was similar to the response of the calves to the purchased lubricant, the two groups will be discussed together. The first symptom noticed in all calves was lacrimation, which began from the 5th to the 13th day and generally lasted throughout the duration of the disease. In some cases the lacrimation was so profuse that the cheeks remained wet. A slight rise in temperature (103.0°F. to 104.6°F.) lasting for a few days, occurred in four calves about the time of the appearance of lacrimation. All calves showed slight loss of appetite during the early phases of symptoms. A transient diarrhea appeared in three calves from the 5th to the 11th day, and in two calves in the later stages, on the 27th and 29th days. Four calves showed some salivation, which began as early as the 9th day and as late as the 18th day.

Red areas on the lips and hard palate occurred in all calves. These areas appeared on the 8th day after beginning the feeding in one case and as late as the 22nd day in another. These areas disappeared in some instances, to be followed by development of similar areas at other sites. The gums (Fig. 1) of four calves became red and swollen. This change began at one incisor and in some cases progressed until the entire gum was involved. Papilloma-like proliferative changes developed in each calf either on the tongue, gums, dental pad, or muzzle (Figs. 2, 3, 4, 5). One calf developed such changes at all of these sites and a second calf developed them at three of the areas.

The skin of all calves exposed to the lubricant became dry, hard and thickened. The folds of the skin became especially prominent over the face, neck and shoulders. These changes began as early as the 11th day from the initial feeding in one calf and as late as the 29th day in another (Figs. 5, 7).

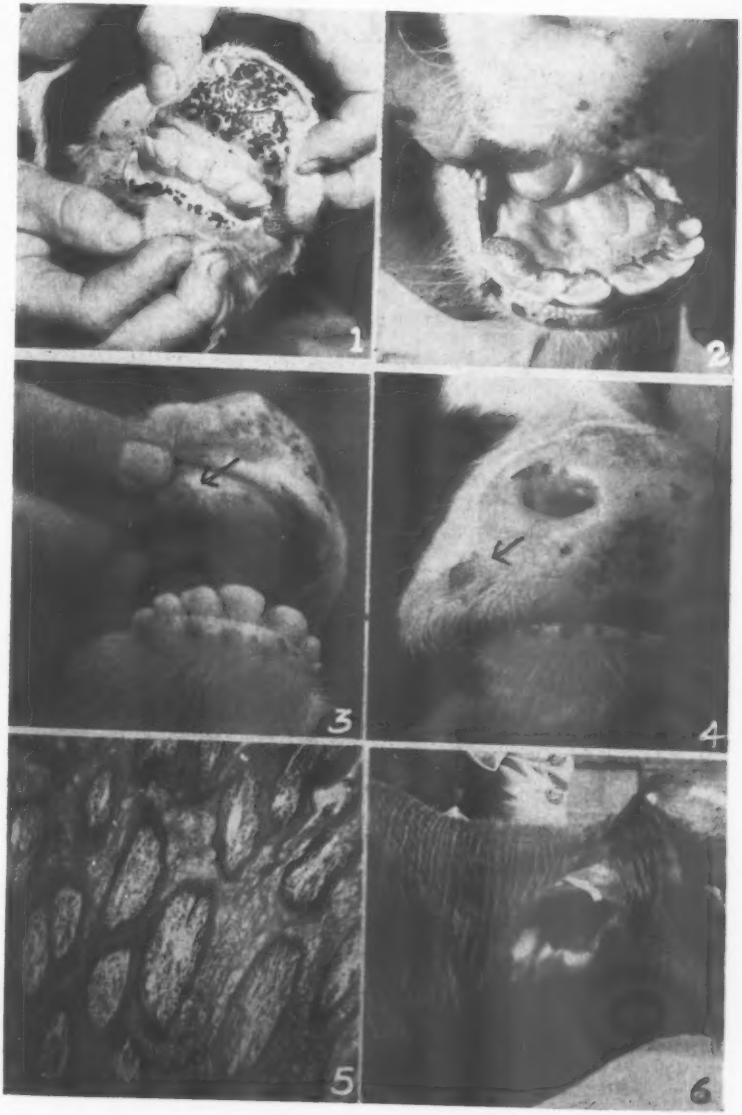
The same general picture was presented by all calves on autopsy. In addition to the skin and proliferative changes already described, gross changes were found in the digestive and urinary tracts of the calves. Flat ulcers were present in the abomasum of 5 calves; in some instances the mucosa of the intestinal tract was thickened. The kidneys appeared enlarged and pale, and pale streaks were present in the cortex. Each liver presented light colored areas, varying in size, on the surface; one liver had an extra-hepatic cyst. These areas extended into the parenchyma. The gall bladder was distended in most calves, and the wall appeared thickened in all cases. Small cyst-like nodules on the lining of the gall bladder were seen in 5 calves.

Examination of tissues obtained at autopsy revealed histological changes that are in agreement with the changes reported by the authors previously cited. An accumulation of keratinized material and keratinization of the hair follicles developed (Fig. 8) although these changes were not so extensive as those seen in more chronic cases. Cystic dilatations of the collecting tubules in the kidney cortex with accompanying fibrosis were present in all calves (Fig. 9). Areas of degenerating liver cells (Fig. 10) with occasional cellular infiltrations, were seen in some liver sections. Some areas of fibrosis were also present, as were proliferations of the bile ducts. Cystic dilatations of the glands of the gall bladder were observed in some instances (Fig. 11). These occurred at the site of the nodules observed at autopsy. The kidney changes were uniform for all calves; however, the other changes described do not represent a complete and detailed study of all individuals.

The cause of the condition produced by the lubricant was rapid when compared to spontaneous cases of hyperkeratosis. One calf, No. 1351, developed a secondary abscess of the tongue and pharynx and died on the 25th day. The remainder of the calves were allowed to go until they became extremely weak, at which time autopsies were performed. The duration, under these conditions, ranged from 27 to 39 days. The amount of lubricant administered varied from 70 grams administered in 17 days, to 255 grams administered in 18 days. There was no recognizable relation of severity of the symptoms produced to the amount administered. The initial symptoms of lacrimation appeared in Calf 252 in 7 days and after a total of 30 grams of lubricant while in Calf 1351 lacrimation appeared on the 11th day after a total of 150 grams of lubricant. There appeared no relation between the gross changes produced and the degree of exposure, except possibly in Calf 252. This calf, which received the smallest amount, developed the most typical proliferative changes of the oral mucosa. This suggests that the administration of smaller amounts of the lubricant might result in a more prolonged and chronic syndrome. The cessation of administration of the lubricant resulted in no improvement in the condition of the calves, and it is apparent that the amounts given were more than sufficient to bring about the symptoms and lesions observed. The time of appearance of the first symptoms, subsequent developments, and the duration of the disease produced are in general agreement with the data reported by McEntee, Hansel, and Olafson (1951).

LEGEND FOR FIGURES

Fig. 1—Gums reddened and swollen, Calf 247, at 21 days. Fig. 2—Oral papillomas, Calf 252, at 25 days. Fig. 3—Papilloma on dental pad, Calf 252, at 25 days. Fig. 4—Wart-like proliferation of muzzle, Calf 252, at 25 days. Fig. 5—Section of proliferative lesion mouth, Calf. 252. Fig. 6—Hyperkeratosis, Calf 242, at 36 days.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SUMMARY

Hyperkeratosis has been experimentally produced in calves by the administration of a lubricant recovered from a farm on which the disease had occurred and by a similar product purchased on the market. Typical symptoms and lesions were produced in six calves while two controls, one of which was in contact exposure, remained normal. As little as 70 grams administered over a period of 17 days produced the symptoms and lesions. As far as the author is aware this is the first experimental production of bovine hyperkeratosis with a petroleum product.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Dr. S. G. Guss, Lynchburg, Va., whose interest and assistance in obtaining the materials have been invaluable; to Drs. H. K. Royer and G. J. Fleck, Lynchburg, Va., for their interest; to Dr. H. S. Mosby and Mr. G. H. Brown, Biol. Dept., V.P.I., for the photographs; to Miss B. V. Conner, Biol. Dept., V.P.I., for the photomicrographs and technical assistance; to the V.P.I. Animal and Dairy Husbandry Depts. for support; and to Dr. A. M. Lee, U.S.D.A., B.A.I., who examined the data presented and the manuscript.

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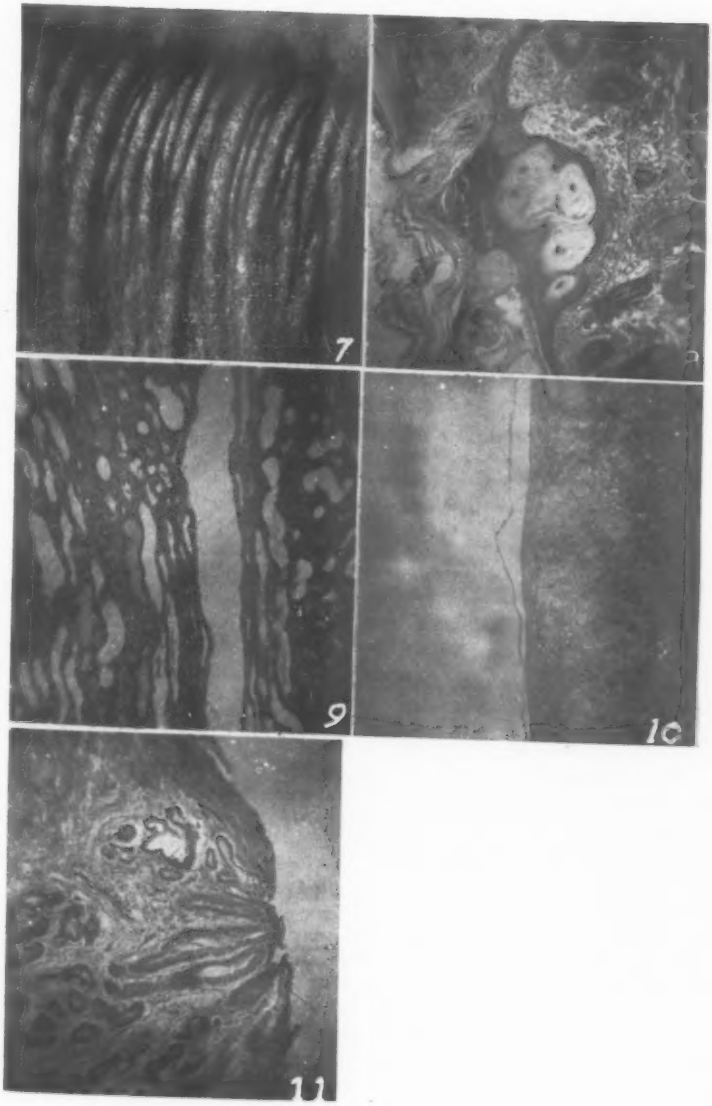
LEGEND FOR FIGURES

Fig. 7—Skin area of neck, Calf 1371, at 38 days. Fig. 8—Section of skin, Calf 1366, showing keratinized layer and keratinization of hair follicles. Fig. 9—Section of kidney, Calf 242, showing cystic dilatation of tubules. Fig. 10—Section of liver, Calf 242, showing degeneration of cells. Fig. 11—Section of gall bladder, Calf 1366, showing dilatation of glands.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News and Notes

(*Editor's note*: *News contributions should be sent to the person whose name appears at the end of the appropriate section.*)

NOTES FROM THE PRESIDENT

I have a few brief announcements and comments for which I beg a moment of your attention.

Your Council, at the request of the Chamberlin Hotel at Old Point Comfort, has agreed to postpone our annual meeting to the third week-end in May, 1952, that is, the 15th, 16th, and 17th. This was done because the original date agreed upon proved difficult for the hotel management We hope this will not inconvenience you and that you will make your plans accordingly.

May I call attention to the notices which you will receive from your Section Secretaries in February calling for papers for the May meetings. Please cooperate with the deadline which he sets for receiving these titles for he must make up your Section program and submit it to our Editor by March 5th. At the same time your Section Secretary will request you to submit an abstract of your paper not to exceed 200 words. Please give your attention to this matter because if it is too long, the Editorial Staff will find it impractical to consult with you individually and will be forced to resort to their own judgment with regard to its reduction.

At the suggestion of Mr. John Thornton Wood, Chairman of our Committee on Resource-Use Education, I wrote Governor John S. Battle requesting him to examine the possibility of making available to the public schools literature on Conservation in a form that would be of value to these groups. He referred this matter to the Department of Education and it is our hope that steps in this direction will be taken.

The Administrative Secretary of the American Association for the Advancement of Science requested the members of the Association's Council to write to the Senate Appropriations Committee to restore the budget of the National Science Foundation reduced by the House Committee from the requested fourteen million dollars to three hundred thousand dollars. Nearly thirteen millions of the requested budget had been earmarked for basic research and for a fellowship program. I wrote in the name of our Academy to the Chairman of the Senate Appropriations Committee. I am recently informed that budget was finally approved at three million dollars. I regret that the letters of scientists did not move the Congressmen to restore more of the requested budget.

Dr. Henry Leidheiser, Jr. has written a fine article in which he tries to interest more industrial scientists in our State in the Virginia Academy of Science and in presenting more papers at our annual meet-

ings. (See The Bulletin of the Virginia Section of the American Chemical Society, Vol. 29, numbers 1 and 3, 1951.) I wish to extend a cordial invitation to all of these scientists not yet members to join this Academy and take an active part on its programs.

Finally, I would request you to send me such suggestions as you deem appropriate and feasible of execution designed to make our Academy function more effidiently for the cause of scientific research, science teaching, and for the development of interest in science in Virginia.

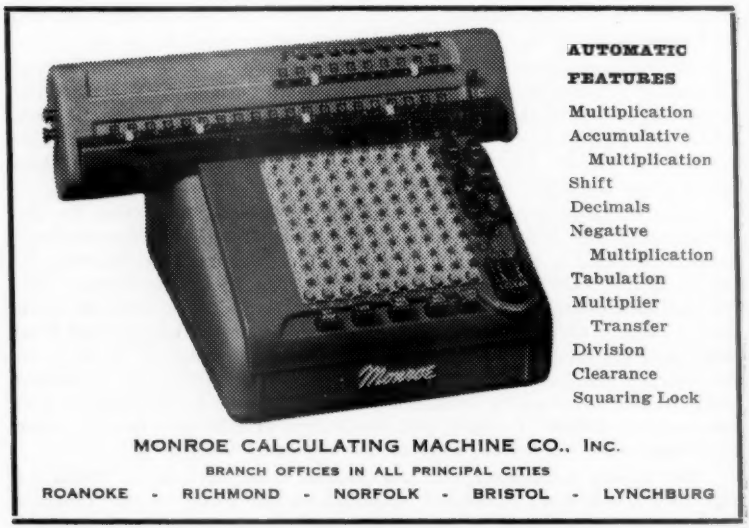
PAUL M. PATTERSON

RESEARCH COMMITTEE

Encouragement of research in Virginia has always been one of the chief objectives of the Academy. Research Grants-in-Aid, and also the annually available J. Shelton Horsley Research Award, are offered by the Academy to encourage and stimulate research work of its members. The attention of Academy members is called to the availability of both the Grants-in-Aid, and the Award. The Research Committee urges the greater interest and participation of members in both.

RESEARCH GRANTS-IN-AID

The income from the Academy's Endowment Fund is set aside each year to provide assistance to Academy members who submit to the



Research Committee worthy research problems for which financial aid is needed. Research grants have varied in amount but during recent years have averaged approximately $100.00 each.

The applicant is requested to outline his problem and list the items for which assistance is required. The names and addresses of three persons who know his qualifications for scientific research should be submitted with the request.

Grants will be made on the basis of (1) apparent merit of the research project, and (2) the need of the grant to aid in carrying out a project, or some portion of it. Worthy problems being worked on by younger members, and by members on staffs of smaller schools—as well as investigations by older members and by those on staffs of larger institutions—will be given full consideration in the awarding of grants.

Since funds are now available, the committee urges that applications be mailed at an early date to W. S. Flory, Jr., Boyce, Virginia, so that they may be considered at its next meeting.

J. SHELTON HORSLEY RESEARCH AWARD

Members of the Academy and officers of Sections are requested to give thought to the entering of papers for the 1952 Horsley Research Award. The only requirements for eligibility for the J. Shelton Horsley Research Award in 1951 were that submitted papers should (1) present original research (2) by Academy members (3) at the annual May meeting, - (4) the paper dealing either with unpublished data or be a published work - if publication had not been prior to May of the previous year.

It is anticipated that essentially the same rules will apply to eligibility in 1952 as in 1951. Details of these will be worked out by the Research Committee early in 1952, and will be supplied to every member by the secretaries of the various Sections with their calls for papers for the May meeting. Members desiring details of eligibility before receiving the secretaries call for papers will be furnished with this information or inquiry to the Research Committee.

The Section Officers are again asked to serve as advisors to the Research Committee and to pass on to it papers which they deem prize-worthy, even though they are not formally submitted for the award. This procedure in 1950 and 1951 resulted in a marked increase in the number of papers considered.

The Horsley Award is one of the top prizes available to scientists in Virginia. The widespread interest and participation of members in the seeking of this award will mark a healthy condition with respect to the type of research work being carried out in the state. Such interest and participation is desired and solicited.—W. S. FLORY, JR., THE BLANDY EXPERIMENTAL FARM, UNIVERSITY OF VIRGINIA, BOYCE

INSTITUTE OF TEXTILE TECHNOLOGY

Eleven Research Fellows arrived at the Institute of Textile Technology, Charlottesville, on September 13 to begin their first year in the Institute's two-year graduate program.

Their arrival marks the fifth academic year of the Institute educational program which leads to the Master of Science degree in textile technology. Only students showing a special interest and aptitude in textile science or having practical experience in textile work are accepted by the Institute.

The new students are: Richard P. Barber, Bradford Durfee Technical Institute: Anibal L. Ferreira, New Bedford Textile Institute; Isaac Jarkowsky, Brooklyn College; Lawrence C. Legere New Bedford Textile Institute; Earl J. Nickerson, Lowell Textile Institute; Arthur J. Pendleton, Rhode Island School of Design; Robert G. Seid, Georgia Institute of Technology; Robert E. Staples, Bradford Durfee Technical Institute; Albin F. Turbak, New Bedford Textile Institute; James W. Whitworth, Lowell Textile Institute; Theodore Williamson, Jr., Bradford Durfee Technical Institute.

AGRICULTURAL SCIENCE SECTION

Agricultural workers in Virginia regret to hear that Mr. V. A. Tiedjens has resigned as Director of the Virginia Truck Experiment Station to accept a position with a fertilizer concern in Ohio. Mr. Tiedjens has served the vegetable growers faithfully and well during his years at the Norfolk Station. Although the horticultural industry in Virginia is suffering a real loss in the resignation of Mr. Tiedjens all his friends join in wishing him the best of success in his new work in Ohio.

The new Director at the Virginia Truck Experiment Station will be Mr. William H. Brittingham who has been horticulturist at the Texas Agricultural and Mechanic Arts College since 1944. Mr. Brittingham is a plant breeder by training. He was born in Portsmouth, Virginia in 1907, secured his B.S. from Penn State and an A.M. degree from Princeton in 1929 His Ph. D. degree was obtained at Maryland in 1942, with a major in plant breeding.

Mr. Flood Andrews has been granted a leave of absence from his duties in the vegetable section of the department of horticulture at Virginia Polytechnic Institute to assist in the Foreign Agricultural Program of the U.S. Department of Agriculture. Mr. Andrews will be stationed in Bolivia, South America. He will assist in setting up research and demonstration projects in vegetable production.—W. P. JUDKINS, VIRGINIA AGRICULTURAL EXPERIMENT STATION, BLACKSBURG.

ASTRONOMY, MATHEMATICS, PHYSICS SECTION

A new building to house the departments of Mathematics and Physics is under construction at Randolph-Macon College. It is to be

completed at the end of the present session.

Mr. E. E. Gardner former instructor in Physics at Virginia Polytechnic Institute has accepted a position as Assistant Professor of Physics at the Naval Academy.

Mr. John Linsley has joined the staff of Rouss Physical Laboratory as Assistant Professor of Physics. He has recently been associated with Professor E. P. Ney of the University of Minnesota, former member of this section, in cosmic ray research.

Mr. A. R. Kuhlthau has returned to the University of Virginia as a member of the staff of the Naval Ordnance Laboratory. He has been on the Physics staff at the University of New Hampshire.

Mr. J. L. Young, III has resigned from the staff of the Naval Ordnance Laboratory at the University of Virginia to accept a position at Los Alamos.

Mr. R. E. Alley, Jr., formerly at University of Richmond, is now a member of the research department of Bell Laboratories. Professor J. J. Taylor has been named Acting Chairman of the Physics Department in his place.

New members of the Physics staff at University of Richmond are G. P. Williams, Jr. and C. O. Alley, Jr. Mr. Williams was a member of the Carson-Newman faculty before coming to Richmond. Mr. Alley returns from Princeton University where he was doing graduate work.

Professor B. V. English of the Physics Department of Randolph-Macon College has been granted a year's leave of absence to pursue graduate studies at Pennsylvania State College.

Professor J. W. Beams of the University of Virginia has been appointed a member of the committee of the National Heart Institute.

Three new members of the Physics staff at Virginia Military Institute are M. J. Clingan, D. R. Carpenter, and J. B. Breazeale. Mr. Clingan was last year a member of the faculty of Newberry College. Mr. Carpenter comes from Cornell University and Mr. Breazeale from the University of Alabama.—I. G. FOSTER, *Virginia Military Institute.*

BACTERIOLOGY SECTION

The fall meeting of the Bacteriology section was held at the College of William and Mary on November 17, 1951. After a luncheon at the Williamsburg Lodge the following scientific program was presented.

A Simple Medium for the Detection of Pseudomonas, J. M. Sharpley. Laboratories, Dept. of Health, City of Richmond, Virginia.

The Isolation of Enterococci from Urine, A. L. Rosensweig and Norman Ikari. McGuire Veterans Hospital, Richmond, Virginia.

Types of Streptococci found in the Vagina of Mares, Melahat Pusat and P. Arne Hansen. Dept. of Bacteriology and Live Stock Sanitary Service Laboratory, University of Maryland, College Park, Maryland.

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A Possible Source of Error in the Microscopic Examination of Smears for Tuberculosis, A. L. Rosensweig. McGuire Veterans Hospital, Richmond, Virginia.

Streptococci as Secondary Invaders in Blackhead Liver Lesions, Arthur P. Harrison and P. Arne Hansen. Dept. of Bacteriology and Live Stock Sanitary Service Laboratory, University of Maryland, College Park, Maryland.

Fermentation of Inositol by Propionibacterium pentosaceum, Wesley A. Volk. Dept. of Microbiology, University of Virginia, School of Medicine, Charlottesville, Virginia.

Factors Relative to Growth of Endamoeba hystolitica, E. Clifford Nelson. Dept. of Bacteriology and Parasitology, Medical College of Virginia, Richmond, Virginia.

Infectious Hazards of Bacteriological Techniques, Film by U. S. Public Health Service. C.D.C.

IN MEMORIAM

With the death of Doctor George McLean Lawson on September 20, 1951, the members of the Bacteriology section of the Virginia Academy of Science lost one of their most highly esteemed members. Doctor Lawson's leadership was instrumental in the development of this section of the Academy and with his passing the members will deeply miss a fine friend and wise counselor.

—J. DOUGLAS REID, MEDICAL COLLEGE OF VIRGINIA.

BIOLOGY SECTION

Mr. Austin B. Clark's manuscript entitled "The Butterflies of Virginia" has gone to the printer and should appear before 1951. There will be 31 plates illustrating each of the 155 species known from Virginia, and few that should occur here but which have not as yet been found.

A grant-in-aid by the Virginia Academy in 1941 has led to the recent appearance of the first genethnics manual by E. S. Craighill Handy and Elizabeth Green Handy. "The Genethnic Screen" is published by Genethnics Incorporated, Oakton, Virginia.

Mr. Colgate W. Darden, Jr., President of the University of Virginia, has recently announced the promotion of Mr. Edwin M. Betts from Associate Professor to Professor of Biology and Mr. B. F. D. Runk from Assistant to Associate Professor of Biology.

Mr. Dabney Lancaster, President of Longwood College, has announced the promotion of Mr. Robert T. Brumfield from Associate Professor to Professor of Biology.

The following biologists from Virginia worked during the summer at Oak Ridge National Laboratory:

Mr. Robert T. Brumfield
Mr. James Norman Dent

Mr. Orland E. White, Director of the Blandy Experimental Farm, Boyce, Virginia has returned to the University of Virginia after serving the past year as Visiting Professor at University College, Mandalay, Burma.—LADLEY HUSTED, UNIVERSITY OF VIRGINIA

CHEMISTRY SECTION

Frank Berton Carpenter. who was for fifty years Chief Chemist of the Virginia-Carolina Chemical Corporation, died on October 3, 1951, at the age of 87. He was one of the organizers of the Virginia Chemists' Club, in 1907, and was later one of the chairmen of the Virginia Section of the American Chemical Society.

Mr. Earl K. Fischer, former chief of the paint section of the National Bureau of Standards, who for four years was head of the physical chemistry division of the Institute of Textile Technology, at Charlottesville, died of a heart attack at his home in Kensington, Maryland, in August, 1951.

From the Medical College of Virginia: Mr. Charles C. Clayton. assistant professor of biochemistry, has been awarded a grant of $5,184.00 by the United States Public Health Service to apply to research on the relation of minerals to cancer. An additional $4,500.00 is ear-marked for the continuation of this study in its second year.

Mr. Paul S. Larson, research professor, of biochemistry, and also

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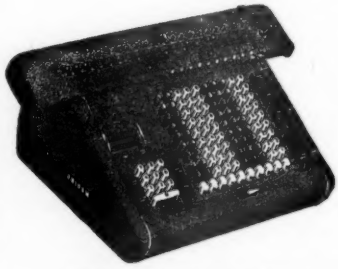

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pharmacologist for the American Tobacco Company Research Laboratory, presented a paper at the American Chemical Society Jubilee in New York in September on "Present Knowledge Concerning the Metabolism of Nicotine by the Animal Body and the Nature and Action of Tobacco Smoke Irritants." Later in September he presented a related paper at the World Tobacco Congress in Amsterdam, Holland.

Mr. R. Blackwell Smith, dean of the School of Pharmacy and associate professor of pharmacology, presented a paper entitled "Toxicological Studies on the Zinc and Disodium Salts of Ethylene-*bis*-dithiocarbamates" at the XIIth International Congress of Pure and Applied Chemistry in New York City in September. Associated with Mr. Smith in this work were Messrs. J. K. Finnegan, P. F. Sahyoun, H. B. Haag, and Paul S. Larson.

The July number of the *Proceedings of the Society for Experimental Biology and Medicine* contained a paper by Messrs. G. Watson James, III, and Lynn D. Abbott, Jr., on the "Effectiveness of Thiocyanate Analog of Vitamin B_{12} in Pernicious Anemia."

Mr. J. C. Forbes and Mr. G. M. Duncan are authors of a paper on "The Effect of Acute Alcohol Intoxication on the Adrenal Glands of Rats and Guinea Pigs," appearing in the September issue of the *Quarterly Journal of Studies on Alcohol.*

From the Virginia Institute for Scientific Research: Mr. Carl J. Likes, former head of the department of chemistry at Hampden-Sydney College, and Mr. Victor J. Kehrer, formerly of the Nickel Fellowship at the Mellon Institute, have joined the staff of the Institute.

From the University of Richmond: Mr. Thomas C. Franklin (Ph. D., Ohio State University) is assistant professor of chemistry in charge of physical chemistry. Mr. Lemuel F. Smith, professor emeritus of Kalamazoo College, is visiting lecturer in chemistry. Puryear Fellowships for graduate study in chemistry have been awarded to Mr. Fletcher B. Owen, from the University of Richmond, and Mr. Ray D. Sothern, from Ursinus College.

From the University of Virginia: Mr. Fred W. Young has returned from Oak Ridge as research associate in chemistry. Mr. Lewis B. Johnson is research associate in chemistry at the School of Engineering. Mr. Tom Crowell has returned from du Pont to his position at the University. Mr. John Cathcart has joined the Division of Metallurgy at Oak Ridge. Mr. Carl D. Lunsford has been awarded a predoctoral du Pont Fellowship for work under the direction of Mr. Robert E. Lutz.

From the Virginia Polytechnic Institute: Under the direction of Mr. Nelson F. Murphy, a listing of "Chemical and Chemical Engineering Journals Available in Virginia" has been completed.

Mr. Frank A. Vingiello spent the summer studying moderators for the Aircraft Nuclear Reactor at the Oak Ridge National Laboratory.

Mr. Russell E. Leed has been granted leave of absence for a year for work with the Atomic Energy Commission at Oak Ridge. Col. W. L. McPherson participated in the 23rd Radioisotopes Techniques Course during the summer at Oak Ridge.

From Hollins College: Miss Lois Ann Trafton (A. B., Hollins) of Norfolk has joined the staff of the department of chemistry.

—WILLIAM E. TROUT, JR., BOX 216,
UNIVERSITY OF RICHMOND, VIRGINIA

ENGINEERING SECTION

The Central Virginia Engineers' Club is an organization of people engaged in or interested in the profession and practice of Engineering, bound in fellowship by the desire for increased professional unity and social relaxation.

Its membership is made up principally from residents of the Richmond, Petersburg, and Hopewell areas. Meetings are held jointly with one of the National Societies such as American Society of Mechanical Engineers, American Society of Civil Engineers, American Society of Refrigerating Engineers, American Institute of Electrical Engineers, American Institute of Architects, and American Chemical Society. In addition to the meetings several inspection trips have been made. The membership is approximately four hundred.

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The Department of Chemical Engineering, Virginia Polytechnic Institute is carrying on research on the electrodeposition of aluminum, financed by a grant from the Air Material Command.

Fred W. Bull, Associate Professor of Chemical Engineering, Virginia Polytechnic Institute, has been on leave, for work at Oak Ridge.

V. G. Szebehely, Applied Mechanics Department, Virginia Polytechnic Institute., resigned in September, 1951, for a research position with the David Taylor Model Basin.

Dan Frederick returned to Virginia Polytechnic Institute after a year's leave of absence at the University of Michigan, where he completed all requirements for his Ph. D. in Mechanics except his thesis. This he is now working on. He was promoted to an Associate Professor in Applied Mechanics in September, 1951.

D. H. Pletta, Head of Applied Mechanics, Virginia Polytechnic Institute, attended the annual meeting of the ASCE in New York, October 22-26, for the dedication ceremonies of the newly formed Engineering Mechanics Division. He is Chairman of the Technical Committee on Experimental Analysis and Analogues.

Mr. Pletta also attended the annual convention of the Virginia Society of Professional Engineers at Old Point Comfort on November 1-2. He is First Vice President of this society for the coming year, and is also a member of the Committee on Education of the National Society of Professional Engineers.

The Virginia Society of Professional Engineers, Inc., Mr. John A. Rives, President, has started publication of a magazine of news and articles relating to professional engineering. Called *The Virginia Engineer,* the journal is published quarterly under Editor H. H. Roberts.

The Virginia Society of Professional Engineers held their third annual meeting at Old Point Comfort on Nov. 2-3, 1951. The program included talks by Louis Lee Guy, Member of the Virginia House of Delegates on "Is Legislation Desirable"; by T. Keith Legare, Executive Secretary of the National Council of State Boards of Engineering Examiners, on "The Development of Engineering Registration"; and by L. L. Dresser, President of the National Society of Professional Engineers, on "Professional Aspects of Members and the National Society."

Mr. Frank C. Vilbrandt, Head of the Department of Chemical Engineering at Virginia Polytechnic Institute, was an Executive Staff member of the Temporary National Office of Ordnance Research, Department of the Army, located at Duke University, Durham, North Carolina, from August 13 to September 15, and is being retained as consultant of the permanent organization, but he is now back in Blacksburg.

Mr. N. F. Murphy, Research Professor in Chemical Engineering, was appointed as delegate to the World Metallurgical Congress in Detroit, Michigan, October 14 to 20, where he was a member of a group of

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international scientists assembled for the exchange of information. Mr. Murphy was assigned to the specific group which discussed the various problems in the field of metallurgical education.

—NELSON F. MURPHY, VIRGINIA POLYTECHNIC INSTITUTE

PSYCHOLOGY SECTION

A number of State psychologists attended the annual meetings of the American Psychological Association in Chicago, August 30 - September 5. From the University of Virginia Mr. Frank A. Geldard spoke at a symposium: Relations between Governmental and Non-Governmental Development of Psychological Resources, and was chairman of one of the sessions sponsored by the Division of Military Psychology, of which he is president. Mr. Richard H. Henneman presented a paper, A Comparison of Vision and Audition as Sensory Channels for Communication. Mr. L. Starling Reid read a paper, The Development of Non-Continuity Behavior through Continuity Learning. Several psychologists from the Lynchburg State Colony also attended the APA meetings: Mr. John N. Buck, Miss Hannah S. Davis, Mr. Allen Cohen, and Mr. William Dakos. Mr. Buck was chairman of a round table: Theoretical and Practical Aspects of the H-T-P Technique; Miss Davis presented a paper, The Case of the Rays.

Mr. John N. Buck retired in September as chief psychologist of the Lynchburg State Colony. His present address is P. O. Box 157, White Stone, Lancaster County, Virginia. Miss Hannah S. Davis was appointed acting chief psychologist to succeed Mr. Buck at the Colony. Miss Lois Brinkmann, of Miami, Florida, was appointed as a psychological interne at the Colony on October 1.

At a meeting of the State Board for the Examination of Clinical Psychologists held at the Lynchburg State Colony on August 4, Mr. Reuben Horlick was certified as a clinical psychologist in the State of Virginia.

The Advisory Psychophysiology Panel of the Office of Naval Research met at the University of Virginia on November 12 and 13. Mr. Frank A. Geldard, a member of the panel and director of a Naval research contract on tactual vibratory sensitivity at the University was host to the group.

The Section Secretary has been notified by the Executive Secretary of the American Psychological Association of the elimination of the membership class of State Affiliate. This action by the Council of Representatives of the APA will in effect discontinue the status of membership in the APA of several members of the Psychology Section of the Virginia Academy of Science. The membership status of State Affiliates was terminated as of December 31, 1951, along with the attendant privilege of subscribing to APA journals at the special rates. Those individuals interested in becoming associate members of the APA may obtain infor-

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mation relative to qualifications and procedures from either Mr. S. B. Williams of the College of William and Mary, or from the Executive Offices of the American Psychological Association, 1515 Massachusetts Avenue, N. W., Washington 5, D. C.

A course in the Rorschach method is being planned at the University of Virginia through the Department of Education for the Spring Semester. The course will be given in three terms, the first year being devoted to basic principles of scoring and interpretation, with the third term being concerned with advanced interpretation. It will be open to practicing clinicians and psychologists with advanced graduate training. For further information applicants may write to Lindley J. Stiles, Dean, Department of Education, University of Virginia, or to the Instructor, Arthur J. Bachrach, Director, Division of Clinical and Medical Psychology, University of Virginia Hospital, Charlottesville, Virginia.

—RICHARD H. HENNEMAN, UNIVERSITY OF VIRGINIA.

STATISTICS SECTION

Mr. Charles Mottley, Mr. J. W. Youden, Mr. Boyd Harshbarger, Mr. R. A. Bradley, Mr. G. W. Tyler, Mr. D. B. Duncan, Mr. M. E. Terry attended the joint Institute of Mathematical Statistics and American Society for Quality Control meetings in celebration of the 50th Anniversary of the National Bureau of Standards. Mr. D. B. Duncan presented a paper, and Mr. J. W. Youden an invited address.

The Department of Statistics of Virginia Polytechnic Institute presented as its fall quarter invited speakers: Mr. John W. Tukey, Princeton, November 2 and 3; Mr. John H. Nair, Director of Research of Thomas Lipton Company, November 26; Mr. D. B. DeLury, Ontario Research Council, December 7 and 8. The department has fourteen graduate students with Mr. Thomas Russell, teaching fellow and Mr. Clyde Kramer in charge of the Statistical Laboratory. Two semi-annual reports on research on statistical methods for sensory difference testing are now available from the Department of Statistics of Virginia Polytechnic Institute.

Mr. Charles Beasley, Fellow of the Institutes of Public Health has resigned his fellowship on the activation of his reserve commission, United States Army.

Mr. E. L. Cox is now on the staff at Dugway Proving Ground, Utah.

The Institute of Mathematical Statistics and the E.N.A.R. section of the Biometrics Society will hold a joint Spring meeting March 19-21, 1952, on the campus of Virginia Polytechnic Institute. Dormitory facilities will be provided for male graduate students for $1.00 per night.

Mr. Elie Weeks and Mr. John W. Griswold, Quartermaster Board, Fort Lee visited the Department of Statistics on Oct. 9. Mr. Griswold has joined the Academy.

Members of the section are invited and encouraged to present papers at the May meeting of the Academy. Titles should be sent to the secretary no later than February 18, 1952.—MILTON E. TERRY, VIRGINIA POLYTECHNIC INSTITUTE.